Séminaires & Congrès

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SELF-SIMILAR PROCESSES AND THEIR APPLICATIONS

Numéro 28

July 20-24, 2009. Angers, France

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
28, 2012, p. 65–87

BASIC PROPERTIES OF THE MULTIVARIATE FRACTIONAL BROWNIAN MOTION

by

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Abstract. — This paper reviews and extends some recent results on the multivariate fractional Brownian motion (mfBm) and its increment process. A characterization of the mfBm through its covariance function is obtained. Similarly, the correlation and spectral analyses of the increments are investigated. On the other hand we show that (almost) all mfBm's may be reached as the limit of partial sums of (super)linear processes. Finally, an algorithm to perfectly simulate the mfBm is presented and illustrated by some simulations.

Résumé **(Propriétés du mouvement brownien fractionnaire multivarié).** — Cet article constitue une synthèse des propriétés du mouvement brownien fractionnaire multivarié (mBfm) et de ses accroissements. Différentes caractérisations du mBfm sont présentées à partir soit de la fonction de covariance, soit de représentations intégrales. Nous étudions aussi les propriétés temporelles et spectrales du processus des accroissements. D'autre part, nous montrons que (presque) tous les mBfm peuvent être atteints comme la limite (au sens de la convergence faible) des sommes partielles de processus (super)linéaires. Enfin, un algorithme de simulation exacte est présenté et quelques simulations illustrent les propriétés du mBfm.

1. Introduction

The fractional Brownian motion is the unique Gaussian self-similar process with stationary increments. In the seminal paper of Mandelbrot and Van Ness [**MandVN68**], many properties of the fBm and its increments are developed (see also [**Taq**] for a review of the basic properties). Depending on the scaling factor

2010 Mathematics Subject Classification. — 26A16, 28A80, 42C40.

Key words and phrases. — Self similarity ; Multivariate process ; Long-range dependence ; Superlinear process ; Increment process ; Limit theorem.

Research supported in part by ANR STARAC grant and a fellowship from Région Rhône-Alpes (France) and a Marie-Curie International Outgoing Fellowship from the European Community.
Research supported in part by ANR InfoNetComaBrain grant.

(called Hurst parameter), the increment process may exhibit long-range dependence, and is commonly used in modeling physical phenomena. However in many fields of applications (e.g. neuroscience, economy, sociology, physics, etc), multivariate measurements are performed and they involve specific properties such as fractality, long-range dependence, self-similarity, etc. Examples can be found in economic time series (see [DavidHash], [FlemYHJ], [GilAlana]), genetic sequences [AriCar09], multipoint velocity measurements in turbulence, functional Magnetic Resonance Imaging of several regions of the brain [AchaBMB08].

It seems therefore natural to extend the fBm to a multivariate framework. Recently, this question has been investigated in [LPS2009, LPS2010, CAA10]. The aim of this paper is to summarize and to complete some of these advances on the multivariate fractional Brownian motion and its increments. A multivariate extension of the fractional Brownian motion can be stated as follows :

Definition 1. — *A Multivariate fractional Brownian motion (p-mfBm or mfBm) with parameter $H \in (0,1)^p$ is a p-multivariate process satisfying the three following properties*

- *Gaussianity,*
- *Self-similarity with parameter $H \in (0,1)^p$,*
- *Stationarity of the increments.*

Here, self-similarity has to be understood as joint self-similarity. More formally, we use the following definition.

Definition 2. — *A multivariate process $(X(t) = (X_1(t), \cdots, X_p(t)))_{t\in\mathbb{R}}$ is said self-similar if there exists a vector $H = (H_1, \cdots, H_p) \in (0,1)^p$ such that for any $\lambda > 0$,*

$$(X_1(\lambda t), \cdots, X_p(\lambda t))_{t\in\mathbb{R}} \stackrel{\text{fidi}}{=} \left(\lambda^{H_1} X_1(t), \cdots, \lambda^{H_p} X_p(t)\right)_{t\in\mathbb{R}}, \tag{1}$$

where $\stackrel{\text{fidi}}{=}$ denotes the equality of finite-dimensional distributions. The parameter H is called the self-similarity parameter.

This definition can be viewed as a particular case of operator self-similar processes by taking diagonal operators (see [DidierPipiras, HudsonMason, LahaRoh, MaejimaMAson]).

Note that, as in the univariate case [Lamperti62], the Lamperti transformation induces an isometry between the self-similar and the stationary multivariate processes. Indeed, from Definition 2, it is not difficult to check that $(Y(t))_{t\in\mathbb{R}}$ is a p-multivariate stationary process if and only if there exists $H \in (0,1)^p$ such that its Lamperti transformation $(t^{H_1} Y_1(\log(t)), \ldots, t^{H_p} Y_p(\log(t)))_{t\in\mathbb{R}}$ is a p-multivariate self-similar process.

The paper is organized as follows. In Section 2, we study the covariance structure of the mfBm and its increments. The cross-covariance and the cross-spectral density of the increments lead to interesting long-memory type properties. Section 3 contains the time domain as well as the spectral domain stochastic integral representations of the mfBm. Thanks to these results we obtain a characterization of the mfBm through its covariance matrix function. Section 4 is devoted to limit theorems, the mfBm is obtained as the limit of partial sums of linear processes. Finally, we discuss in Section 5 the problem of simulating sample paths of the mfBm. We propose to use the Wood and Chan's algorithm [**WoodC94**] well adapted to generate multivariate stationary Gaussian random fields with prescribed covariance matrix function.

2. Dependence structure of the mfBm and of its increments

2.1. Covariance function of the mfBm. — In this part, we present the form of the covariance matrix of the mfBm.

Firstly, as each component is a fractional brownian motion, the covariance function of the i-th component is well-known and we have

$$\mathbb{E}X_i(s)X_i(t) = \frac{\sigma_i^2}{2}\left\{|s|^{2H_i} + |t|^{2H_i} - |t-s|^{2H_i}\right\}. \tag{2}$$

with $\sigma_i^2 := \mathrm{var}(X_i(1))$. The cross covariances are given in the following proposition.

***Proposition 3* (Lavancier *et al.* [LPS2009]).** — *The cross covariances of the mfBm satisfy the following representation, for all $(i,j) \in \{1,\dots,p\}^2$, $i \neq j$,*

1. *If $H_i + H_j \neq 1$, there exists $(\rho_{i,j}, \eta_{i,j}) \in [-1,1]\times\mathbb{R}$ with $\rho_{i,j} = \rho_{j,i} = \mathrm{corr}(X_i(1), X_j(1))$ and $\eta_{i,j} = -\eta_{j,i}$ such that*

$$\begin{aligned}\mathbb{E}X_i(s)X_j(t) = \frac{\sigma_i\sigma_j}{2}\Big\{&(\rho_{i,j} + \eta_{i,j}\mathrm{sign}(s))|s|^{H_i+H_j} + (\rho_{i,j} - \eta_{i,j}\mathrm{sign}(t))|t|^{H_i+H_j} \\ &-(\rho_{i,j} - \eta_{i,j}\mathrm{sign}(t-s))|t-s|^{H_i+H_j}\Big\}.\end{aligned} \tag{3}$$

2. *If $H_i + H_j = 1$, there exists $(\tilde\rho_{i,j}, \tilde\eta_{i,j}) \in [-1,1]\times\mathbb{R}$ with $\tilde\rho_{i,j} = \tilde\rho_{j,i} = \mathrm{corr}(X_i(1), X_j(1))$ and $\tilde\eta_{i,j} = -\tilde\eta_{j,i}$ such that*

$$\mathbb{E}X_i(s)X_j(t) = \frac{\sigma_i\sigma_j}{2}\left\{\tilde\rho_{i,j}(|s| + |t| - |s-t|) + \tilde\eta_{i,j}(t\log|t| - s\log|s| - (t-s)\log|t-s|)\right\}. \tag{4}$$

Proof. — Under some conditions of regularity, Lavancier *et al.* [**LPS2009**] actually prove that Proposition 3 is true for any L^2 self-similar multivariate process with stationary increments. The form of cross covariances is obtained as the unique solution of a functional equation. Formulae (3) and (4) correspond to expressions given in [**LPS2009**] after the following reparameterization : $\rho_{i,j} = (c_{i,j} + c_{j,i})/2$ and $\eta_{i,j} = (c_{i,j} - c_{j,i})/2$ where $c_{i,j}$ and $c_{j,i}$ arise in [**LPS2009**].

□

Remark 1. — *Extending the definition of parameters $\rho_{i,j}, \tilde{\rho}_{i,j}, \eta_{i,j}, \tilde{\eta}_{i,j}$ to the case $i = j$, we have $\rho_{i,i} = \tilde{\rho}_{i,i} = 1$ and $\eta_{i,i} = \tilde{\eta}_{i,i} = 0$, so that (2) coincides with (3) and (4).*

Remark 2. — *The constraints on coefficients $\rho_{i,j}, \tilde{\rho}_{i,j}, \eta_{i,j}, \tilde{\eta}_{i,j}$ are necessary but not sufficient conditions to ensure that the functions defined by (3) and (4) are covariance functions. This problem will be discussed in Section 3.4.*

Remark 3. — *Note that coefficients $\rho_{i,j}, \tilde{\rho}_{i,j}, \eta_{i,j}, \tilde{\eta}_{i,j}$ depend on the parameters (H_i, H_j). Assuming the continuity of the cross covariances function with respect to the parameters (H_i, H_j), the expression (4) can be deduced from (3) by taking the limit as H_i+H_j tends to 1, noting that $((s+1)^H - s^H - 1)/(1-H) \to s\log|s| - (s+1)\log|s+1|$ as $H \to 1$. We obtain the following relations between the coefficients : as $H_i + H_j \to 1$*

$$\rho_{i,j} \sim \tilde{\rho}_{i,j} \qquad \textit{and} \qquad (1 - H_i - H_j)\eta_{i,j} \sim \tilde{\eta}_{i,j}.$$

This convergence result can suggest a reparameterization of coefficients $\eta_{i,j}$ in $(1 - H_i - H_j)\eta_{i,j}$.

2.2. The increments process. — This part aims at exploring the covariance structure of the increments of size δ of a multivariate fractional Brownian motion given by Definition 1. Let $\Delta_\delta X = (X(t+\delta) - X(t))_{t\in\mathbb{R}}$ denotes the increment process of the multivariate fractional Brownian motion of size δ and let $\Delta_\delta X_i$ be its i-th component.

Let $\gamma_{i,j}(h,\delta) = \mathbb{E}\Delta_\delta X_i(t)\Delta_\delta X_j(t+h)$ denotes the cross-covariance of the increments of size δ of the components i and j. Let us introduce the function $w_{i,j}(h)$ given by

$$w_{i,j}(h) = \begin{cases} (\rho_{i,j} - \eta_{i,j}\mathrm{sign}(h))|h|^{H_i+H_j} & \text{if } H_i + H_j \neq 1, \\ \tilde{\rho}_{i,j}|h| + \tilde{\eta}_{i,j} h \log|h| & \text{if } H_i + H_j = 1. \end{cases} \tag{5}$$

Then from Proposition 3, we deduce that $\gamma_{i,j}(h,\delta)$ is given by

$$\gamma_{i,j}(h,\delta) = \frac{\sigma_i\sigma_j}{2}\Big(w_{i,j}(h-\delta) - 2w_{i,j}(h) + w_{i,j}(h+\delta)\Big). \tag{6}$$

Now, let us present the asymptotic behaviour of the cross-covariance function.

Proposition 4. — *As $|h| \to +\infty$, we have for any $\delta > 0$*

$$\gamma_{i,j}(h,\delta) \sim \sigma_i\sigma_j\delta^2|h|^{H_i+H_j-2}\kappa_{i,j}(\mathrm{sign}(h)), \tag{7}$$

with

(8)

$$\kappa_{i,j}(\mathrm{sign}(h)) = \begin{cases} (\rho_{i,j} - \eta_{i,j}\mathrm{sign}(h))(H_i + H_j)(H_i + H_j - 1) & \textit{if } H_i + H_j \neq 1, \\ \tilde{\eta}_{i,j}\mathrm{sign}(h) & \textit{if } H_i + H_j = 1. \end{cases}$$