Quantum Hyberboloid and Braided Modules

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Abstract

We construct a representation theory of a "quantum hyperboloid" in terms of so-called braided modules. We treat these objects in the framework of twisted Quantum Mechanics.

Résumé

Nous construisons une théorie de représentations pour « l'hyperboloïde quantique » en termes de modules tressés. Nous traitons ces objets dans le cadre de la mécanique quantique tordue.

1 Introduction

In the present paper we study a quantum hyperboloid from the point of view of the generalized framework for quantum mechanics suggested in [GRZ]. The main idea of that paper is the following. Quantizing a degenerate Poisson bracket we have, in general, to modify the ordinary notions of quantum mechanics, namely, those of Lie algebra, trace and conjugation (involution) operators.

Meanwhile, all objects and operators disscused in [GRZ] were connected to an involutive $S^2 = id$ solution to the quantum Yang-Baxter equation (QYBE)

$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}.$$

In particular, such objects arise as a result of a quantization of some Poisson brackets (P.b.) generated by a skew-symmetric ($R \in \wedge^2(g)$) solution to the classical Yang-Baxter equation (CYBE),

$$[[R, R]] = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0,$$

where g is a Lie algebra. Another family of examples of such a type of objects is related to non-quasiclassical (or, in other words, non-deformational) solutions of the QYBE, cf. [G1], [GRZ].

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More precisely, given a representation $\rho : g \to Vect(M)$ of a Lie algebra g in the space of vector fields on a manifold or algebraic variety M, then the bracket

$$\{f, g\}_R = \mu < \rho^{\otimes 2}(R), df \otimes dg >, \ f, g \in Fun(M)$$

is Poisson. Hereafter μ denotes the product in the algebra under question and $< \, , \, >$ denotes the pairing between the vector fields and the differential forms extended on their tensor powers. Quantizing this Poisson bracket, we get an algebra belonging to a twisted, i.e., equipped with a Yang-Baxter twist S, tensor symmetric category ("symmetric" means that this twist is involutive). Moreover, this algebra is S-commutative, i.e., the product μ in it satisfies the relation $\mu = \mu S$.

Thus, by deforming the commutative algebra $Fun(M)$ "in the direction" of the above P.b. we get a S-commutative algebra. It is more interesting to deform in a similar way the non-commutative algebras, for example, those arising from a quantization of the Kirillov-Kostant-Souriau (KKS) bracket on a given coadjoint orbit in g^*.

Let us assume that $\rho = ad^*$. Then the bracket $\{ \, , \, \}_R$ is well defined on g^* as well as on any orbit in g^*. It is not difficult to see that the KKS bracket and this "R-matrix bracket" $\{ \, , \, \}_R$ are compatible and this problem can be thought of as one of simultaneous quantization of the whole of the Poisson pencil (P.p.) generated by these two brackets.

In this connection the following question arises: what is a quantization of the KKS bracket? There exist (at least) two ways to represent the quantum objects. On the one hand it is possible to think of these objects as the quotient algebras of the enveloping algebras $U(g)_h$ (h means here that this parameter is introduced as a factor in the Lie bracket in the definition of the enveloping algebra).

On the other hand the quantum object can be represented into $End(V)$ where V is a suitable Hilbert space. Such a representation can be constructed by means of a geometric quantization method or by means of an orbit method, but in numerous cases both approaches provide similar results.

We treat the algebra structure arising from the quantization of the KKS bracket in one or in other way, and we are interested in a further deformation of this algebra. In what follows the latter procedure will be called "twisting" to distinguish the two types of quantization. Roughly speaking, a twisting is a passage to a twisted category instead of the "classical" one. When a twisting arises from the above mentioned solutions of the CYBE, it can be performed by means of an operator $F = F_\nu$ (represented as a series in a parameter ν) such that $S = F^{-1}\sigma F$ where σ denotes the flip. Existence of such a series F has been established by V.Drinfeld in [D].

As a result, the principal objects and operators of the ordinary quantum mechanics can be twisted by means of F_ν. In particular, a usual trace becomes

S-commutative, i.e., such that $tr(A \circ B) = tr \circ S(A \otimes B)$ where $\circ$ denotes the operator product. A Lie bracket turns into an S-Lie bracket in the sense of [G1], [GRZ]. etc.

Our principal aim is to generalize this approach to the case when R is a solution of the modified CYBE. This means that the above element $[[R, R]]$ is g-invariant. In this case the R-matrix bracket is Poisson only on certain orbits in g^* which are called, according to the terminology of [GP], *the orbits of R-matrix type*. However, if $g = sl(2)$, all orbits in g^* are of the R-matrix type.

The result of the quantization of the above P.p. on a given orbit in g^* can be represented as a three parameter algebra $A^c_{h,q}$ where h is a parameter of quantization of the KKS bracket, q a parameter of twisting and c labels the orbits. $c = 0$ corresponds to the cone.

The algebras of such type have been considered in plenty of papers. We refer the reader to [P] where these algebras (equipped with a traditional involution) have appeared under the name of "quantum spheres" (see the discussion of involutions in Section 5).

It was shown in [DG1] that these algebras represent flat deformations of their classical counterparts. In this paper we realize the second step of the quantization procedure and develop a representation theory for the algebras $A^c_{h,q}$ in terms of *braided modules*.

Roughly speaking, a braided module is a $U_q(g)$-module equipped with a representation $\rho : A^c_{h,q} \to End(V)$ in such a way that the map ρ is a $U_q(g)$-morphism.

In this sense we treat the triple $(A^c_{h,q}, V, \rho)$ as an object of twisted quantum mechanics (more precisely, of the particular case, connected to the quantum group $U_q(g)$). In the present paper we consider the simplest example of such twisted quantum mechanics, namely, the one connected to the quantum hyperboloid and its modules.

Although an axiomatic approach to such a version of quantum mechanics has not yet been adequately developed, it is clear that the traditional involution approach is not reasonable for such a type of objects, since the maps of these algebras into $End(U_k)$, where U_k are the braided modules mentioned above, do not respect such an involution. In the present paper we suggest another way to coordinate the involution with a braided structure.

The paper is organized as follows. In the next section we recall the constructions of [DG1]. In Section 3 we develop a representation theory for this algebra in terms of braided modules. In Section 4 we consider the so-called braided Casimir, i.e., an invariant (with respect to the action of the quantum group) element and assign to it operators acting in braided modules. We prove that the latter operators are scalar, and we compute the eigenvalues of the braided Casimir. The last section is devoted to a discussion of the braided (twisted) traces and involutions as ingredients of twisted quantum mechanics.

Throughout the paper $U_q(g)$-Mod will denote the category of $U_q(g)$-modules. We include in it, besides the finite-dimensional modules, their inductive limits. The parameter q is assumed to be generic, and the basic field k is C or R (in the latter case we consider the normal form of the Lie algebra g).

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2 Quantum hyperboloid: basic notions

To construct a quantum hyperboloid it is sufficient to fix a representation of the quantum group $U_q(sl(2))$ into a three dimensional space V, decompose the space $V^{\otimes 2}$ into a direct sum of irreducible $U_q(sl(2))$-modules and impose a few natural equations on elements of $V^{\otimes 2} \oplus V \oplus k$ which are compatible with the action of the quantum group $U_q(sl(2))$ and are similar to their classical counterparts.

Thus, let us consider the algebra $U_q(sl(2))$ generated by the elements H, X, Y satisfying the well-known relations

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

Let us equip this algebra with a coproduct defined on the basic elements in the following way

$$\Delta(X) = X \otimes 1 + q^{-H} \otimes X,\ \Delta(Y) = 1 \otimes Y + Y \otimes q^H,\ \Delta(H) = H \otimes 1 + 1 \otimes H.$$

It is well-known that this algebra has a Hopf structure, being equipped with the antipode γ defined by

$$\gamma(X) = -q^H X,\ \gamma(H) = -H,\ \gamma(Y) = -Yq^{-H}.$$

Let us consider a linear space V with the base $\{u, v, w\}$, and turn V into a $U_q(sl(2))$-module by setting

$$Hu = 2u,\ Hv = 0,\ Hw = -2w,\ Xu = 0,\ Xv = -(q + q^{-1})u,\ Xw = v,$$

$$Yu = -v,\ Yv = (q + q^{-1})w,\ Yw = 0.$$

It is easy to show that the above relations for H, X, Y are satisfied.

We want to stress that throughout this paper we deal with a coordinate representation of module elements. We consider the endomorphisms as matrices and their action as left-multiplication by these matrices.

Using the coproduct we can equip $V^{\otimes 2}$ with a $U_q(sl(2))$-module structure as well. This module is reducible and can be decomposed into a direct sum of three irreducible $U_q(sl(2))$-modules

$$V_0 = \mathrm{span}((q^3+q)uw + v^2 + (q+q^{-1})wu),$$

$$V_1 = \mathrm{span}(q^2uv - vu,\ (q^3+q)(uw-wu) + (1-q^2)v^2,\ -q^2vw + wv),$$

$$V_2 = \mathrm{span}(uu,\ uv + q^2vu,\ uw - qvv + q^4wu,\ vw + q^2wv,\ ww)$$

of spins 0, 1 and 2 (hereafter the sign $\otimes$ is omitted).

Then only the following relations imposed on the elements of the space $V^{\otimes 2}\oplus V\oplus k$ are compatible with the $U_q(sl(2))$-action:

$$C_q = (q^3+q)uw + vv + (q+q^{-1})wu = c,\ q^2uv - vu = -2hu,$$

$$(q^3+q)(uw-wu) + (1-q^2)v^2 = 2hv,\ -q^2vw + wv = 2hw$$

with arbitrary h and c. The element C_q will be called a *braided Casimir*.

Therefore it is natural to introduce a *quantum hyperboloid* as the quotient algebra of the free tensor algebra $T(V)$ over the ideal generated by elements

$$(q^3+q)uw + v^2 + (q+q^{-1})wu - c,\ q^2uv - vu + 2hu,$$

$$(q^3+q)(uw-wu) + (1-q^2)v^2 - 2hv,\ -q^2vw + wv - 2hw.$$

This quotient algebra will be denoted by $A^c_{h,q}$.

The quotient algebra of $T(V)$ over the ideal generated by the latter three elements will be denoted by $A_{h,q}$. This algebra is another (compared with the quantum algebra $U_q(sl(2))$) q-analogue of the enveloping algebra $U(sl(2))$.

In [DG2] it has been shown that both algebras $A^c_{h,q}$ and $A_{h,q}$ represent the flat deformations of their classical counterparts. Let us make some comments on the proof.

Concerning the algebra $A^c_{h,q}$, the proof of flatness is based on the two following statements:

1. The algebra $A^0_{0,q}$ is Koszul (see [BG] for definition). This fact was proved in [DG1] "by hands". Now there exists (for the case $q = 1$ and hence for a generic q since the deformation $A^0_{0,1} \to A^0_{0,q}$ is flat) a more conceptual proof valid for any simple Lie algebra (see [Be], [Bo]).