Orbits of Matrix Tuples

Lieven Le BRUYN*

Abstract

In this paper we outline a procedure which can be seen as an approximation to the well known "hopeless" problem of classifying m-tuples ($m \geq 2$) of $n \times n$ matrices under simultaneous conjugation by GL_n. The method relies on joint work with C. Procesi, on the étale local structure of matrix-invariants and recent work [10], [11] on the nullcone of quiver-representations.

Résumé

Dans ce papier, nous présentons une procédure qui peut être considérée comme une approximation au problème bien connu et "sans espoir" de classification des m-uplets ($m \geq 2$) de matrices $n \times n$ sous l'action de conjugaison simultanée par GL_n. La méthode est basée sur un travail en commun avec C. Procesi, sur la structure étale locale des invariants de matrices et sur un travail récent de l'auteur sur le cône nilpotent des représentations de carquois.

Throughout, we fix an algebraically closed field of characteristic zero and call it $\mathbb{C}$. Let $X_{m,n} = M_n(\mathbb{C})^{\oplus m}$ be the affine space of m-tuples of $n \times n$ matrices with the action of GL_n given by simultaneous conjugation, that is

$$g.X = g.(x_1, \ldots, x_m) = (gx_1g^{-1}, \ldots, gx_mg^{-1})$$

for all $g \in GL_n$ and all $X \in X_{m,n}$. The first approximation to the orbit space of this action is the quotient variety $V_{m,n}$ which is determined by its coordinate ring which is the ring of invariant polynomial functions $\mathbb{C}[V_{m,n}] = \mathbb{C}[X_{m,n}]^{GL_n}$. The inclusion $\mathbb{C}[V_{m,n}] \subset \mathbb{C}[X_{m,n}]$ gives the quotient map

$$\pi : X_{m,n} \longrightarrow V_{m,n} = X_{m,n}//GL_n$$

Procesi [14] has shown that the coordinate ring $\mathbb{C}[V_{m,n}]$ is generated by traces in the generic matrices of degree at most n^2. From general invariant theory [13] we know that the points of $V_{m,n}$ classify the closed orbits in $X_{m,n}$. The correspondence being given by associating to a point $\zeta \in V_{m,n}$ the orbit of minimal dimension in the fiber $\pi^{-1}(\zeta)$.

AMS 1980 *Mathematics Subject Classification* (1985 *Revision*): 16R30, 16G20

*Senior researcher at the NFWO — Departement Wiskunde UIA, B-2610 Wilrijk (Belgium)

A more algebraic interpretation is as follows. A point $X = (x_1, \dots, x_m) \in X_{m,n}$ determines an n-dimensional representation of $\mathbb{C} < X_1, \dots, X_m >$ by associating to X the algebra map

$$\varphi_X : \mathbb{C} < X_1, \dots, X_m > \mapsto M_n(\mathbb{C})$$

given by $X_i \mapsto x_i$. Two representations φ_X and φ_Y are isomorphic if and only if X and Y belong to the same orbit. By the Artin-Voigt theorem [6, II.2.7] the closed orbits correspond to the semi-simple n-dimensional representations. A general orbit is mapped under the quotient π to its semi-simplification, that is the direct sum of the Jordan-Hölder components.

Our first aim is to study the highly singular variety $V_{m,n}$ better. Assume that $\zeta \in V_{m,n}$ determines a semi-simple n-dimensional representation of the form

$$S_1^{\oplus e_1} \oplus \dots \oplus S_r^{\oplus e_r}$$

where the S_i are the distinct simple components of dimension k_i occuring with multiplicity e_i. We then say that ζ is of representation type

$$\tau = (e_1, k_1; \dots; e_r, k_r)$$

where this tuple is of course only determined upto permuting the indices. The algebraic notion of degeneration of representation types can be described combinatorially as follows. We say that $\tau' = (e'_1, k'_1; \dots; e'_{r'}, k'_{r'}) < \tau$ if there is a permutation σ on $\{1, \dots, r'\}$ such that there exist numbers

$$1 = j_0 < j_1 < j_2 < \dots < j_r = r'$$

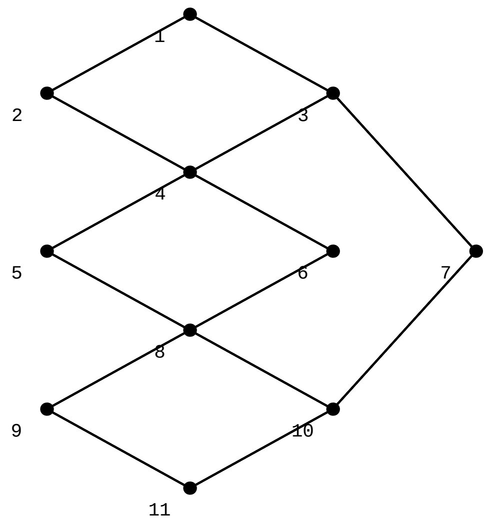
such that for every $1 \leq i \leq r$ we have

- $e_i k_i = \sum_{j=j_{i-1}+1}^{j_i} e'_{\sigma(j)} k'_{\sigma(j)}$
- $e_i \leq e_{\sigma(j)'}$ for all $j_{i-1} < j \leq j_i$

For example for $n = 3$ we have 5 representation types with a line degeneration pattern $(3,1) < (2,1;1,1) < (1,1;1,1;1,1) < (1,2;1,1) < (1,3)$. However, things quickly become more complex. For $n = 4$ we have 11 representation types

type	τ
1	$(1, 4)$
2	$(1, 3; 1, 1)$
3	$(1, 2; 1, 2)$
4	$(1, 2; 1, 1; 1, 1)$
5	$(1, 1; 1, 1; 1, 1; 1, 1)$
6	$(1, 2; 2, 1)$
7	$(2, 2)$
8	$(1, 1; 1, 1; 2, 1)$
9	$(1, 1; 3, 1)$
10	$(2, 1; 2, 1)$
11	$(4, 1)$

with corresponding Hasse diagram



With $V_{m,n}(\tau)$ we will denote the set of points ζ of $V_{m,n}$ of representation type τ. An application of the Luna slice theorem ([12] and [17]) gives the following result. The crucial observation in the proof is that the representation type determines the conjugacy class of the isotropy group of the corresponding closed orbit.

Proposition 1 ([8, II.1.1]) — *$\{V_{m,n}(\tau); \tau\}$ is a finite stratification of $V_{m,n}$ into locally closed irreducible smooth algebraic subvarieties.*

$V_{m,n}(\tau')$ lies in the Zariski closure of $V_{m,n}(\tau)$ if and only if $\tau' < \tau$.

Further, one can use the theory of trace identities [15] to describe the defining equations of these locally closed subvarieties $V_{m,n}(\tau)$. Thus, we may assume that we have a firm grip on these strata. Remains the difficulty of studying the orbit structure of the fiber $\pi^{-1}(\zeta)$ for ζ in a fixed stratum $V_{m,n}(\tau)$. That is, we want to describe the isomorphism classes of n-dimensional representations with a fixed Jordan-Hölder decomposition. Again, the first step is provided by the Luna slice machinary.

Let $X \in X_{m,n}$ be a point lying on the unique closed orbit in the fiber $\pi^{-1}(\zeta)$ for $\zeta \in V_{m,n}(\tau)$. Let G_X denote the isotropy group, then the tangent space at X, $T_X(X_{m,n}) \simeq X_{m,n}$ has a splitting as G_X-modules as

$$T_X(X_{m,n}) = T_X(GL_n.X) \oplus N_X$$

where $T_X(GL_n.X)$ is the tangent space to the orbit and N_X the corresponding normal space. By the Luna slice theorem we have in a neighborhood of $0 \in N_X$ the following commutative diagram

$$\begin{array}{ccc} GL_n \times^{G_X} N_X & \xrightarrow{\alpha} & X_{m,n} \\ \downarrow & & \downarrow \\ N_X//G_X & \xrightarrow{\alpha'} & V_{m,n} \end{array}$$

where α is determined by sending the class of (g,n) to $g.(X+n)$, where $GL_n \times^{G_X} N_X = (GL_n \times N_X)//G_X$ under the action $h.(g,n) = (gh^{-1}, h.n)$ and where both α and α' are étale maps.

It follows from this description (see for example [17, p.101]) that the fiber at ζ is isomorphic to

$$\pi^{-1}(\zeta) \simeq GL_n \times^{G_X} \mathrm{Null}(N_X, G_X)$$

as GL_n-varieties where we denote by $\mathrm{Null}(N_X, G_X)$ the nullcone of the G_X-action on the normalspace N_X, that is, if

$$N_X \xrightarrow{\pi'} N_X//G_X$$

the nullcone $\mathrm{Null}(N_X, G_X) = \pi'^{-1}(\pi'(0))$. In particular, we deduce that the orbit structure of the fibers $\pi^{-1}(\zeta)$ is the same along a stratum $V_{m,n}(\tau)$ and is fully understood provided we know the G_X-orbit structure in the nullcone $\mathrm{Null}(N_X, G_X)$.

In order to achieve this goal we need to have a better representation theoretic description of the normal space N_X, of the isotropy group G_X and of its action on N_X. These facts can best be described in terms of quiver representations. Let us recall some definitions.

A **quiver** Q is a 4-tuple (Q_v, Q_a, t, h) where Q_v is a finite set $\{1, ..., k\}$ of vertices, Q_a a finite set of arrows φ between these vertices and $t, h : Q_a \to Q_v$ are two maps assigning to an arrow φ its tail t_φ and its head h_φ respectively. Note that we do not exclude loops or multiple arrows.

A **representation** V of a quiver Q consists of a family $\{V(i) : i \in Q_v\}$ of finite dimensional $\mathbb{C}$-vector spaces and a family $\{V(\varphi) : V(t_\varphi) \to V(h_\varphi); \varphi \in Q_a\}$ of linear maps between these vectorspaces, one for each arrow in the quiver. The dimension-vector $dim(V)$ of the representation V is the k-tuple of integers $(dim(V(i)))_i \in \mathbb{N}^k$. We have the natural notion of morphisms and isomorphisms between representations consisting of k-tuples of linear maps with obvious commutativity conditions.

For a fixed dimension-vector $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ one defines the **representation space** $R(Q, \alpha)$ of the quiver Q to be the set of all representations V of Q with $V_i = \mathbb{C}^{\alpha_i}$ for all $i \in Q_v$. Because $V \in R(Q, \alpha)$ is completely determined by the linear maps $V(\varphi)$, we have a natural vector space structure

$$R(Q, \alpha) = \underset{\varphi \in Q_a}{\oplus} M_\varphi(\mathbb{C})$$

where $M_\varphi(\mathbb{C})$ is the vector space of all $\alpha_{h_\varphi} \times \alpha_{t_\varphi}$ matrices over $\mathbb{C}$.

We consider the vector space $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and function field $\mathbb{C}(Q, \alpha)$. There is a canonical action of the linear reductive group

$$GL(\alpha) = \prod_{i=1}^{k} GL_{\alpha_i}(\mathbb{C})$$

on the variety $R(Q, \alpha)$ by base change in the V_i. That is, if $V \in R(Q, \alpha)$ and $g = (g(1), ..., g(k)) \in GL(\alpha)$, then

$$(g.V)(\varphi) = g(h_\varphi) V(\varphi) g(t_\varphi)^{-1}$$

The $GL(\alpha)$-orbits in $R(Q, \alpha)$ are precisely the isomorphism classes of representations.

Let us return to our problem of describing the G_X-action on the nullcone $\mathrm{Null}(N_X, G_X)$. To a representation type $\tau = (e_1, k_1; \ldots; e_r, k_r)$ we associate a quiver Q_τ and a dimension vector α_τ in the following way.

- Q_τ is the quiver on r-vertices $\{v_1, \ldots, v_r\}$ with
  - $(m-1)k_i^2 + 1$ loops at vertex v_i
  - $(m-1)k_i k_j$ directed arrows from v_i to v_j
- $\alpha_\tau = (e_1, \ldots, e_r)$