Some Conjectures About Invariant Theory and their Applications

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Abstract

It turns out that various algebraic computations can be reduced to the same type of computations: one has to study the series of integrals $\int_K f^n(k)g(k)\,dk$, where f, g are complex valued K-finite functions on a compact Lie group K. So it is tempting to state a general conjecture about the behavior of such integrals, and to investigate the consequences of the conjecture.

MAIN CONJECTURE: *Let K be a compact connected Lie group and let f be a complex-valued K-finite function on K such that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Then for any K-finite function g, we have $\int_K f^n(k)g(k)\,dk = 0$ for n large enough.*

Especially, we prove that the main conjecture implies the jacobian conjecture. Another very optimistic conjecture is proposed, and its connection to isospectrality problems is explained.

Résumé

Il se trouve que divers calculs algébriques se réduisent à un même type de calcul : il s'agit d'étudier des intégrales $\int_K f^n(k)g(k)\,dk$, où f, g sont des fonctions K-finies et à valeurs complexes sur un groupe de Lie compact K. Il est alors tentant de formuler une conjecture générale sur de telles intégrales et en explorer les conséquences.

CONJECTURE PRINCIPALE : *Soit K un groupe de Lie compact connexe et soit f une fonction K-finie et à valeurs complexes sur K telle que $\int_K f^n(k)\,dk = 0$ pour tout $n > 0$. Alors pour toute fonction K-finie g, on a $\int_K f^n(k)g(k)\,dk = 0$ pour n assez grand.*

En particulier, nous montrons que la conjecture principale implique la conjecture jacobienne. Nous proposons une autre conjecture optimiste et expliquons ses liens avec les problèmes d'isospectralité.

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Introduction

It turns out that various algebraic computations can be reduced to the same type of computations: one has to study the series of integrals $\int_K f^n(k)g(k)\,dk$, where f, g are complex valued K-finite function on a compact Lie group K. So it is tempting to state general conjectures about the behavior of such integrals, and to investigate the consequences of these conjectures. Here we will state the following two conjectures:

Main Conjecture — *Let K be a compact connected Lie group and let f be a complex-valued K-finite function on K such that $\int_K f^n(k)\,dk = 0$ for any $n > 0$. Then for any K-finite function g, we have $\int_K f^n(k)g(k)\,dk = 0$ for n large enough.*

Second Conjecture — *Let $G \supset L$ be a reductive spherical pair, let $f \in \mathbb{C}[G/L]$, and let $C^{\#}$ be the G-complement of $\mathbb{C}$ in $\mathbb{C}[G/L]$. If $f^n \in C^{\#}$ for any $n \geq 1$, then 0 belongs to $\overline{G.f}$.*

These two conjectures are closely related. Indeed the second conjecture implies the main one. In this paper we show various examples of questions which can be treated (or partially solved) by the Conjectures above. The main two examples are as follows:

First example: recall that the Jacobian Conjecture states that a volume-preserving polynomial map $F\colon \mathbb{C}^n \to \mathbb{C}^n$ is invertible. In the paper we show that the Jacobian Conjecture follows the main conjecture (see Sections 2, 3, 4 and 5).

Second example: recall that two smooth real-valued functions f, g defined on a compact riemannian manifold are called isospectral if $\Delta + f$ and $\Delta + g$ have the same spectrum. We will see that some results of isospectral rigidity for $\mathbb{RP}^2$ follows from the second conjecture. It should be noted that the second conjecture and the section 7 has been motivated by Guillemin's paper [G].

In order to give some support to the main conjecture, we will see that the integrals $\int_K f^n(k)g(k)\,dk$ are closely related. Indeed we prove that all formal series $\chi_g = \sum_{n\geq 0}(\int_K f^n(k)g(k)\,dk)\,z^n$ can be deduced from one of them by applying a differential operator, see Section 6. To give some motivation for the second conjecture, we will see that a conjecture about invariant theory due to Guillemin implies a special case of the second conjecture.

At the end of the paper, we will investigate the conjecture when the group is a torus. In this case, the integrals considered appear naturally in the computation of the Hasse invariant and in the computation of number of points modulo p of plane algebraic curves.

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1 Equivalent forms of the conjecture.

In the section, we use the classical correspondence between compact Lie groups and algebraic reductive groups to state three equivalent forms of the main conjecture (see (1.1), (1.3), (1.7)).

Let K be a compact group. A continous complex-valued function defined on K will be called *K-finite* if the K-module generated by f is finite dimensional. Equivalently, f is a matrix coefficient of a finite dimensional representation. Denote by dk the Haar measure of K. The main conjecture of the paper is as follows:

Main Conjecture 1.1 — *Let K be a compact connected Lie group and let f, g be complex-valued K-finite functions. Assume that $\int_K f^n(k)\, dk = 0$ for any $n > 0$. Then $\int_K f^n(k)g(k)\, dk = 0$ for n large.*

Let G be a connected reductive algebraic group over an algebraically closed field F of characteristic zero. Denote by $\hat{G}$ the space of isomorphism classes of simple rational representations of G (for simplicity, the elements in $\hat{G}$ will be called the *types* of G). For any type $\tau \in \hat{G}$, denote by τ^* the dual type. For any G-module M, set $M = \bigoplus_{\tau\in\hat{G}} M_\tau$, where M_τ is the τ-isotypical component of M. Similarly, for any $m \in M$, set $m = \sum_{\tau\in\hat{G}} m_\tau$, where m_τ is the τ-isotypical component of m. In particular, denote by M_{triv} and m_{triv} the trivial components. Also set $X(m) = \{\tau \in \hat{G} | m_\tau \neq 0\}$. We have $F[G]_{\mathrm{triv}} = F$, hence we can define a linear form $L\colon F[G] \to F$ by $L(f) = f_{\mathrm{triv}}$.

Lemma 1.2 — (i) *Assume $F = \mathbb{C}$. Let K be a maximal compact subgroup of G. Then we have $L(f) = \int_K f(k)\, dk$.*

(ii) *The bilinear form $b\colon F[G] \times F[G] \to F$, $f, g \mapsto L(fg)$ is non degenerate.*

Proof. (i) Since K is Zariski dense in G, the map $L'\colon f \in F[G] \mapsto \int_K f(k)\, dk$ is G-invariant. Since $F[G]_{\mathrm{triv}} = F$ and $L'(1) = L(1) = 1$, L' and L are equal.

(ii) Clearly, the kernel of b is a G-invariant ideal of $F[G]$. Hence its zero set in G is G-stable and so the kernel of b is zero. ☐

Let us call *G-algebra* any commutative algebra endowed with a rational action of G by algebra automorphisms. For a G-algebra A, denote by $C(A)$ the conjecture:

$C(A)$: *Let $f \in A$ and $\tau \in \hat{G}$. Assume that $(f^n)_{\mathrm{triv}} = 0$ for all $n > 0$. Then $(f^n)_\tau = 0$ for n large.*

Corollary 1.3 — *Assume that the main conjecture holds. Then the conjecture $C(F[G])$ holds.*

Proof. Let $f \in F[G]$. Note that f is defined over a finitely generated subfield E of F and such a field can be embeded in $\mathbb{C}$. Hence we can assume that $F = \mathbb{C}$. Let $\tau \in \hat{G}$. The τ^*-component of $\mathbb{C}[G]$ is finite dimensional. Let K be a maximal compact subgroup of G. By hypothesis, we have $\int_K f^n(k)\, dk = 0$ for any $n > 0$. By the main conjecture 1.1, there exist $N = N(\tau)$ such that $\int_K f^n(k)g(k)dk = 0$ for all $g \in \mathbb{C}[G]_{\tau^*}$ and $n \geq N(\tau)$. By Lemma 1.2, we have $(f^n)_\tau = 0$ for any $n \geq N(\tau)$. ☐

For X, Y two subsets of $\hat{G}$, denote by $X.Y$ the set of all types occuring in the tensor product $x \otimes y$ for some $x \in X$ and some $y \in Y$.

Lemma 1.4 — *Let A be a G-algebra, let I be a G-invariant nilpotent ideal, let $f \in A$ and let $\overline{f} \in A/I$ be its residue modulo I. There exists an integer $d \geq 0$ and some finite subsets $X_0, X_1, ..., X_d$ in $\hat{G}$ such that $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(\overline{f}^{n-i})$, for any $n \geq d$.*

Proof. Denote by A_0 the algebra A with a trivial action of G. The structure map $\Delta\colon A \to F[G] \otimes A_0$ is an injective morphism of G-algebras. Hence we can assume that A is of the form $F[G] \otimes R$ for some algebra R, and I is of the form $F[G] \otimes J$ for some ideal J of R. Witout loss of generality, we can assume that R is finitely generated and J is the radical of R.

It follows from the existence of a primary decomposition for R that R embeds in a finite sum of primary algebras (apply Theorem 11 of [Ms] to the R-module R). Hence we can assume that $A \simeq F[G] \otimes R$, where R is primary and noetherian, and $I \simeq F[G] \otimes J$, where J is the radical of R. As R embeds in its quotient field, we can assume that R is already a quotient field. By Cohen's structure theorem (Theorem 60 of [Ms]), we have $R \simeq L \oplus J$, where $L \simeq R/J$ is a field. Thus we have $R \otimes F[G] \simeq L \otimes F[G] \oplus J \otimes F[G]$ and accordingly, we have $f = \overline{f} + h$, where $h \in F[G] \otimes J$.

Let d such that $J^{d+1} = 0$, and set $X_i = X(h^i)$. We have $f^n = \sum_{0 \leq i \leq d} \binom{n}{i} h^i.\overline{f}^{n-i}$. Hence we have $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(g^{n-i})$, for any $n \geq d$. ☐

For any $X \subset \hat{G}$, denote by X^* the set of all types dual to those of X. For any $X, Y \subset \hat{G}$, denote by $X : Y$ the set of all types $\mu \in \hat{G}$ such that τ occurs in $\mu \otimes \sigma$ for some $\tau \in X$ and $\sigma \in Y$. For a sequence of subsets X_n in $\hat{G}$, we denote by $\lim X_n$ the set of all $\tau \in \hat{G}$ which belongs to infinitely many X_n. With these notations, the conclusion of conjecture $C(A)$ can be written as $\lim X(f^n) = \emptyset$.

Lemma 1.5 — (i) *Let $X, Y \subset \hat{G}$. We have $X : Y = X.Y^*$.*

(ii) *Let X_n be a sequence of subsets in $\hat{G}$ and let $X \subset \hat{G}$ be finite. Then* $\lim(X_n.X) = (\lim X_n).X$.

Proof. (i) We have $Hom_G(\sigma \otimes \mu, \tau) \simeq Hom_G(\mu, \tau \otimes \sigma^*)$. Hence $X : Y = X.Y^*$.

(ii) Let $\tau \in \lim(X_n.X)$. Hence we have $X_n \cap (\{\tau\} : X) \neq \emptyset$ for infinitely many n. As X is finite, $\{\tau\} : X$ is finite. Hence there exists some $\mu \in \{\tau\} : X$ such that μ belongs to infinitely many X_n. Hence τ belongs to $(\lim X_n).X$, and we have $\lim(X_n.X) \subset (\lim X_n).X$. As the opposite inclusion is obvious, (ii) follows. □

Lemma 1.6 — *Let A be a commutative G-algebra and let I be a G-invariant nilpotent ideal. Then conjecture $C(A/I)$ implies conjecture $C(A)$.*

Proof. Assume $C(A/I)$. Let $f \in A$, let $\overline{f} \in A/I$ be its residue modulo I and let $d \geq 0$, $X_0, ..., X_d \subset \hat{G}$ as in lemma 1.4. Assume that $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$. We have $X(f^n) \subset \cup_{0 \leq i \leq d} X_i.X(\overline{f}^{n-i})$, for any $n \geq d$. Hence by lemma 1.5, we have $\lim X(f^n) = \emptyset$. So conjecture $C(A)$ holds. □

Corollary 1.7 — *Assume the main conjecture. Then for any G-algebra A, the conjecture $C(A)$ holds.*

Proof. Using lemma 1.6, we reduce the conjecture $C(A)$ for a general G-algebra A to the case where A is prime. So we will assume that A is prime. Let Φ be its fraction field. The structure map $\Delta : A \to F[G] \otimes A_0$ (where A_0 is the algebra A with a trivial action of G) induces a G-equivariant embeding $A \to \Phi[G]$. Hence the conjecture $C(A)$ follows from corollary 1.3. □

It is possible to prove a very special case of the main conjecture, namely:

Proposition 1.8 — *Let V be a G-module, let $f \in V$ and let $\tau \in \hat{G}$. Consider f as an element of the G-algebra SV and assume that $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$. Then $(f^n)_\tau = 0$ for n large.*

Proof. There is a natural comultiplication map $\Delta : SV \to SV \otimes SV$ which is dual of the algebra structure on SV^*. For $n \geq 0$, let $B_n \subset S^nV$ be the G-module generated by f^n. We have $\Delta(f^n) = n! \sum_{p+q=n} f^p/p! \otimes f^q/q!$. Thus $\bigoplus_{n \geq 0} B_n$ is a sub-coalgebra of SV. Hence $R = \bigoplus_{n \geq 0} B_n^*$ is a quotient algebra of SV^*. Let $\tau \in \hat{G}$. By Hilbert's Theorem, R_{τ^*} is finitely generated as a R_{triv}-module. As $R_{\mathrm{triv}} = \mathbb{C}$, R_{τ^*} is finite dimensional, i.e. $(f^n)_\tau = 0$ for n large. □

Remark. Let $f \in V$ as in Proposition 1.8. Indeed we have $(f^n)_{\mathrm{triv}} = 0$ for any $n > 0$ if and only if f is in the nilcone of V, i.e. 0 is belongs to the closure of the G-orbit of f.

2 A technical version of the main conjecture

In order to show that the Jacobian conjecture follows from the main conjecture (Section 5), we state another version of the main conjecture (Proposition 2.2).