Dense Orbits in Orbital Varieties in $\mathfrak{sl}_n$

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Abstract

Let $\mathcal{O}$ be a nilpotent orbit in the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and let $\mathcal{V}$ be an orbital variety contained in $\mathcal{O}$. Let $\mathbf{P}$ be the largest parabolic subgroup of $\mathrm{SL}(n,\mathbb{C})$ stabilizing $\mathcal{V}$. We describe nilpotent orbits such that all the orbital varieties in them have a dense $\mathbf{P}$ orbit and show that for n big enough the majority of nilpotent orbits do not fulfill this.

Résumé

Soit $\mathcal{O}$ une orbite nilpotente dans l'algèbre de Lie $\mathfrak{sl}_n(\mathbb{C})$ et soit $\mathcal{V}$ une variété orbitale contenue dans $\mathcal{O}$. Soit $\mathbf{P}$ le plus grand sous-groupe parabolique de $\mathrm{SL}(n,\mathbb{C})$ stabilisant $\mathcal{V}$. Nous décrivons les orbites nilpotentes dont toutes les variétés orbitales contiennent une $\mathbf{P}$-orbite dense et montrons que pour n assez grand la majorité des orbites nilpotentes n'ont pas cette propriété.

1 Introduction

1.1 Let $\mathbf{G}$ be a connected semisimple finite dimensional complex algebraic group. Let $\mathfrak{g}$ be its Lie algebra and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Consider the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. A $\mathbf{G}$ orbit $\mathbb{O}$ in $\mathfrak{g}$ is called nilpotent if it consists of nilpotent elements.

Fix some triangular decomposition $\mathfrak{g} = \mathfrak{n} \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}^-$. An irreducible component of $\mathbb{O} \cap \mathfrak{n}$ is called an orbital variety contained in $\mathbb{O}$. Orbital varieties play a key role in Springer's Weyl group representations and in the primitive ideal theory of $U(\mathfrak{g})$.

The last can be detailed as follows. Since $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. This identification gives an adjoint orbit a symplectic structure. Let $\mathcal{V}$ be an orbital variety contained in $\mathbb{O}$. After N. Spaltenstein [Sp] and

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R. Steinberg [St] one has

$$(*) \qquad \dim \mathcal{V} = {}^{1}\!/_{2} \dim \mathcal{O}.$$

Moreover it was pointed out by A. Joseph [J] that this implies that an orbital variety is a Lagrangian subvariety of its nilpotent orbit. According to the orbit method philosophy one would like to attach an irreducible representation of $U(\mathfrak{g})$ to $\mathcal{V}$. This can be naturally implemented in the case of $\mathfrak{g} = \mathfrak{sl}_n$ where there exists a one to one correspondence between the set of primitive ideals of $U(\mathfrak{g})$ containing the augmentation ideal of its centre and the set of orbital varieties in $\mathfrak{g}$. Moreover as it is shown in [M2] in this case $\overline{\mathcal{V}}$ is the associated variety of the corresponding simple highest weight module so that orbital varieties give a natural geometric understanding of the classification of primitive ideals. Hence the study of orbital varieties in $\mathfrak{sl}_n$ is especially interesting.

1.2 Orbital varieties remain rather mysterious objects. The only general description was given by R. Steinberg [St] and is as follows. Let $R \subset \mathfrak{h}^*$ be the set of roots, R^+ be the choice of positive roots defining $\mathfrak{n}$ and $\Pi \subset R^+$ be the corresponding set of simple roots. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ acting on R. Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ corresponding to the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \bigoplus \mathfrak{n}$. Recall that $\mathfrak{n} = \bigoplus_{\alpha\in R^+} X_\alpha$ (resp.$\mathfrak{n}^- = \bigoplus_{\alpha\in -R^+} X_\alpha$) where X_α is the root subspace corresponding to α. For $w \in W$ set $\mathfrak{n} \cap^w \mathfrak{n} := \bigoplus_{\alpha\in R^+\cap w(R^+)} X_\alpha$. For each subgroup $\mathbf{H}$ of $\mathbf{G}$ let $\mathbf{H}(\mathfrak{n}\cap^w \mathfrak{n})$ be the set of $\mathbf{H}$ conjugates of $\mathfrak{n}\cap^w \mathfrak{n}$. One easily sees that there exists a unique nilpotent orbit $\mathcal{O}$ such that $\mathbf{G}(\mathfrak{n} \cap^w \mathfrak{n}) = \overline{\mathcal{O}}$. Then $\mathcal{V}_w = \overline{\mathbf{B}(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}$ is an orbital variety and the map $\phi : w \mapsto \mathcal{V}_w$ is a surjection of W onto the set of orbital varieties.

This description is not very satisfactory from the geometric point of view since a $\mathbf{B}$ invariant subvariety generated by a linear space is a very complex object. One of the attempts to give a reasonable description of an orbital variety is the following conjecture proposed by S. P. Smith.

Given an orbital variety $\mathcal{V}$ let $\mathbf{P}_{\mathcal{V}}$ be its stabilizer. This is a standard parabolic subgroup of $\mathbf{G}$. We say that an orbital variety $\mathcal{V}$ is of S type if there exists a dense $\mathbf{P}_{\mathcal{V}}$ orbit in it. We say that a nilpotent orbit $\mathcal{O}$ is of S type if all its orbital varieties are of S type.

Conjecture 1.1 (S. P. Smith) — *In $\mathfrak{sl}_n$ all orbital varieties are of S type.*

The truth of this conjecture would give a more elegant and simple description of orbital varieties. For a given orbital variety closure it would provide a way to construct a resolution of its singularities and be the first step towards a description of its ideal of definition. It could also provide a natural way to define orbital varieties in the case of quantum groups. These implications made the conjecture (suggested by S.P. Smith some ten years ago) quite attractive.

1.3 The conjecture is true for $\mathfrak{sl}_n$ when $n \leq 8$ as shown by E. Benlolo in [B]. Yet here we show that the conjecture is false in general.

In 2.5 we give the first counter-example to the Smith conjecture which appears in $\mathfrak{sl}_9$ and is the only counter-example for $n \leq 9$. We give some other counter-examples which we use in what follows.

Further we investigate the situation for $n \gg 0$. In § 3 we give sufficient conditions for an orbit to be not of S type. This can be explained as follows.

Take $\mathfrak{g} = \mathfrak{sl}_n$. Consider $\mathfrak{sl}_i$ for $i < n$ as a Levi subalgebra $\mathfrak{l}_{1,i}$ of $\mathfrak{g}$ (cf. 3.2). Set $\mathfrak{n}_{1,i} = \mathfrak{n} \cap \mathfrak{l}_{1,i}$ and define the projection $\pi_{1,i} : \mathfrak{n} \to \mathfrak{n}_{1,i}$. A result of [M1] is that $\pi_{1,i}$ takes an orbital variety closure in $\mathfrak{sl}_n$ to an orbital variety closure in $\mathfrak{sl}_i$.

Given an orbital variety $\mathcal{V}$ let $\tau(\mathcal{V})$ be its τ-invariant (cf. 2.4). As we explain in 3.2 if $\mathcal{V}$ is of S type and $\alpha_i \notin \tau(\mathcal{V})$ then $\pi_{1,i}(\mathcal{V})$ must be of S type. From this given an orbital variety not of S type in $\mathfrak{sl}_i$ we show how to construct orbital varieties not of S type in $\mathfrak{sl}_n$ for $n > i$.

1.4 In § 4 we give sufficient conditions for an orbit to be of S type. This can be explained as follows.

Orbital varieties are irreducible components of $\mathcal{O}\cap\mathfrak{n}$. Yet they are as far as possible of being disjoint. Indeed after N. Spaltenstein [Sp] for any two orbital varieties $\mathcal{V}, \mathcal{V}' \subset \mathcal{O}$ there exist a chain of orbital varieties $\mathcal{V} = \mathcal{V}_1, \cdots, \mathcal{V}_k = \mathcal{V}' \subset \mathcal{O}$ with $\mathrm{codim}(\mathcal{V}_i \cap \mathcal{V}_{i+1}) = 1$ for all $i \in \{1, 2, \cdots, k-1\}$.

In $\mathfrak{sl}_n$ if a nilpotent orbit is neither regular nor minimal it contains more than one orbital variety. Following A. Joseph we apply Vogan's analysis [V] to orbital varieties. For a given orbital variety $\mathcal{V}$ this defines the orbital variety $\mathcal{T}_{\alpha\beta}(\mathcal{V})$ (cf. 4.2). One has $\mathrm{codim}(\mathcal{T}_{\alpha\beta}(\mathcal{V})\cap\mathcal{V}) = 1$ and for any given pair of orbital varieties $\mathcal{V}, \mathcal{V}' \subset \mathcal{O}$ one may pass from $\mathcal{V}$ to $\mathcal{V}'$ by a sequence of $\mathcal{T}_{\alpha\beta}$ operations. This refines Spaltenstein's result.

In each nilpotent orbit there exists a Bala-Carter component (cf. 4.3). As shown by R. Carter in [C] a Bala-Carter component contains a dense **B** orbit. One can use such orbital varieties and Vogan's analysis to construct other orbital varieties of S type; but this does not lead to all orbital varieties of S type. The problem is that the dimension of $\mathbf{P}_{\mathcal{T}_{\alpha\beta}(\mathcal{V})}$ can differ by more than one from the dimension of $\mathbf{P}_{\mathcal{V}}$ and then we cannot conclude that $\mathcal{V}$ of S type implies $\mathcal{T}_{\alpha\beta}(\mathcal{V})$ of S type. Generally speaking this is the reason that the orbital varieties not of S type appear. However the algorithm we obtain is not decisive; but it helps to construct orbital varieties of S type and to give counter-examples to conjecture 1.1.

To show that a specific nilpotent orbit is of S type we find in it enough orbital varieties with a dense **B** orbit so that applying Vogan's analysis we get all the orbital varieties in the given orbit. These computations compose the main part of § 4 and are technically the most difficult part of the work. A few orbits described at the end of § 4 stay unclassified. These cases apparently require more subtle computations.

2 Counter-examples

Lemma 2.1 — *Fix $w \in W$. If the orbital variety $\mathcal{V}_w$ has a dense $\mathbf{P}_{\mathcal{V}_w}$ orbit $\mathcal{P}$ then*

$$\mathcal{P} \cap (\mathfrak{n} \cap^w \mathfrak{n}) \neq \emptyset.$$

It is convenient to replace $\mathfrak{sl}_n$ by $\mathfrak{g} = \mathfrak{gl}_n$. This obviously makes no difference. Note that the adjoint action of $\mathbf{G} = GL_n$ on $\mathfrak{g}$ is just a conjugation.

Let $\mathfrak{n}$ be the subalgebra of strictly upper-triangular matrices in $\mathfrak{g}$ and $\mathbf{B}$ be the (Borel) subgroup of upper-triangular matrices in $\mathbf{G}$. All parabolic subgroups we consider further are standard, that is contain $\mathbf{B}$.

Let e_{ij} be the matrix having 1 in the ij entry and 0 elsewhere. Set $\Pi := \{\alpha_i\}_{i=1}^{n-1}$. Take $i \leq j$. Then for $\alpha = \sum_{k=i}^{j} \alpha_k$, the root space $X_\alpha = \mathbb{C}e_{i,j+1}$ and the root space $X_{-\alpha} = \mathbb{C}e_{j+1,i}$.

We identify W with the permutation subgroup $\mathbf{S}_n$ of GL_n. For $\alpha \in \Pi$ let s_α be the corresponding fundamental reflection and set $s_i = s_{\alpha_i}$.

Let $[\ ,\ ]$ denote the Lie product on $\mathfrak{g}$ given here by commutation in $\operatorname{End} V$. For a standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ we set $\mathfrak{p} := \operatorname{Lie}\mathbf{P}$ which is a standard parabolic subalgebra of $\mathfrak{g}$, that is contains $\mathfrak{b}$.

Lemma 2.2 — *Take $M \in \mathfrak{g}$ and a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. One has*

$$\dim \mathbf{P}M = \dim[\mathfrak{p}, M].$$

Combining these two lemmas we obtain

Corollary 2.3 — *Fix $w \in W$. The orbital variety $\mathcal{V}_w$ is of S type if and only if for some $M \in \mathfrak{n} \cap^w \mathfrak{n}$ one has*

$$\dim[\mathfrak{p}, M] = \dim \mathcal{V}_w.$$

2.2 Nilpotent orbits in $\mathfrak{sl}_n$ are parameterized by Young diagrams. Orbital varieties are parameterized by standard Young tableaux. Let us explain these parameterizations.

In $\mathfrak{sl}_n$ or $\mathfrak{gl}_n$ each nilpotent orbit $\mathcal{O}$ is described by its Jordan form. A Jordan form in turn is parameterized by a partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \lambda_k > 0)$ of n giving the length of Jordan blocks. We denote by $\mathcal{O}_\lambda$ the nilpotent orbit determined by λ.

It is convenient to represent a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ of n as a Young diagram D_λ, that is an array of k rows of boxes starting on the left with the i-th row containing λ_i boxes. The dual partition $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2 \cdots)$ is defined by setting $\hat{\lambda}_i$ equal to the length of the $i-$th column of the diagram D_λ that is $\hat{\lambda}_i = \sharp\{j : \lambda_j \geq i\}$.

One has (cf. [H] § 3.8)

$$(**) \qquad \dim \mathcal{O}_\lambda = n^2 - \sum_{i=1}^{k} \hat{\lambda}_i^2.$$

Define a partial order on partitions as follows. Given two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \mu_j)$ of n we set $\lambda \geq \mu$ if

$$\sum_{l=1}^{i} \lambda_\ell \geq \sum_{\ell=1}^{i} \mu_\ell, \text{ for all } i = 1, 2, \cdots, k.$$

The following result of M. Gerstenhaber (cf. [H] § 3.10) shows that this order corresponds to inclusion of nilpotent orbit closures:

Theorem 2.4 — *Given two partitions λ and μ of n one has $\lambda \geq \mu$ if and only if $\overline{\mathcal{O}}_\lambda \supset \overline{\mathcal{O}}_\mu$.*

2.3 Given a partition λ of n fill the boxes of D_λ with n distinct positive integers. If the entries increases in rows from left to right and in columns from top to bottom we call such an array a Young tableau. If the numbers in Young tableau form a set of integers from 1 to n we call it standard. Let $\mathbf{T}_n$ be the set of standard Young tableaux of size n.

The shape of a Young tableau T is defined to be a Young diagram from which T was built. It defines a partition of n which we denote $\operatorname{sh} T$.

The Robinson - Schensted correspondence $w \mapsto (Q(w), R(w))$ gives a bijection (see, for example [Kn]) from the symmetric group $\mathbf{S}_n$ onto the pairs of standard Young tableaux of the same shape. By R. Steinberg [St1] for all $w, y \in \mathbf{S}_n$ one has $\mathcal{V}_w = \mathcal{V}_y$ iff $Q(w) = Q(y)$. This parameterizes the set of orbital varieties by $\mathbf{T}_n$. Moreover $\operatorname{sh} Q(w) = \lambda$ if and only if $\mathcal{V}_w$ is contained in $\mathcal{O}_\lambda$.

We set $\mathcal{V}_T := \mathcal{V}_w$ if $Q(w) = T$, $\quad \mathbf{P}_T := \mathbf{P}_{\mathcal{V}_T}$ and $\mathfrak{p}_T := \mathfrak{p}_{\mathcal{V}_T}$.

Let T be some Young tableaux with $\operatorname{sh} T = \lambda = (\lambda_1, \cdots)$. Denote by T_j^i its ij-th entry. If k is the entry T_j^i of T, set $r_T(k) = i$ and $c_T(k) = j$.

For $i : 1 \leq i \leq \hat{\lambda}_1$ set $T^i := (T_1^i, \cdots, T_{\lambda_i}^i)$. This is the ordered set of entries of the i-th row. For each $T \in \mathbf{T}_n$ we define $w_r(T) \in \mathbf{S}_n$ through

$$w_r(T) := \begin{pmatrix} 1 \cdots & \cdots & \cdots n \\ T^{\hat{\lambda}_1} & \cdots & T^1 \end{pmatrix}.$$

By [M3], § 3.2.2 $Q(w_r(T)) = T$.