Polynomial Automorphisms and the Jacobian Conjecture

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Abstract

In this paper we give an update survey of the most important results concerning the Jacobian conjecture: several equivalent descriptions are given and various related conjectures are discussed. At the end of the paper, we discuss the recent counter-examples, in all dimensions greater than two, to the Markus-Yamabe conjecture (Global asymptotic Jacobian conjecture).

Résumé

Dans ce papier nous présentons un rapport actualisé sur les résultats les plus importants concernant la conjecture Jacobienne : plusieurs formulations équivalentes et diverses conjectures connexes sont considérées. A la fin du papier, nous donnons les contre-exemples récents, en toute dimension plus grande que deux, à la conjecture de Markus-Yamabe.

Introduction

The last fifteen years the interest in the study of polynomial automorphisms is growing rapidly. The main motivation behind this interest is the existence of several very appealing open problems such as the tame generators conjecture, some linearization problems and last but not least the Jacobian Conjecture.

The aim of this paper is to give a survey of the Jacobian Conjecture, including the most recent results (up to date).

The paper is divided into three parts. In the first chapter a short survey is given of the most important n-dimensional results concerning the Jacobian Conjecture. In the second chapter we study the Jacobian Conjecture from the viewpoint of derivations and relate it to a conjecture about the kernel of a derivation. It turns out that the cases of dimension two and that of dimension bigger than two are essentially different. Finally in the third chapter we discuss some important problems and indicate how they are related to the Jacobian Conjecture.

AMS 1980 *Mathematics Subject Classification* (1985 *Revision*): 14E09

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Acknowledgement

I like to thank the organizers for the possibility of participating in this perfectly organised conference!

1 The Jacobian Conjecture: a short survey

From calculus everyone knows the classical Rolle theorem:

Theorem 1.1 — *If $F : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^1$-function such that $F(a) = F(b)$ for some $a \neq b$ in $\mathbb{R}$, then there exists a point $z \in \mathbb{R}$ such that $F'(z) = 0$.*

The main question of this paper concerns an attempt to generalise this result in a certain direction. More precisely, let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e. a map of the form

$$(x_1, \dots, x_n) \mapsto (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

where each $F_i \in \mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$, the n variable polynomial ring over $\mathbb{C}$. Furthermore for $z \in \mathbb{C}^n$ put $F'(z) := \det(JF(z))$ where

$$JF = \left(\frac{\partial F_i}{\partial X_j}\right)_{1 \leq i,j \leq n}$$

is the Jacobian matrix over F. Now the main question is:

Question 1.2 — *Let $F(a) = F(b)$ for some $a, b \in \mathbb{C}^n$ with $a \neq b$. Does it follow that $F'(z) = 0$ for some $z \in \mathbb{C}^n$.*

The answer (at this moment) is: we don't know if $n \geq 2$! In fact this question is, as we will show below, a reformulation of the famous Jacobian Conjecture.

Conjecture 1.3 (Jacobian Conjecture ($JC(n)$)) — *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $F'(z) \neq 0$ for all $z \in \mathbb{C}^n$ (or equivalently $\det(JF) \in \mathbb{C}^*$), then F is invertible (i.e. F has an inverse which is also a polynomial map).*

To see that the above Rolle type question is indeed equivalent to the Jacobian Conjecture, we recall the following beautiful result due to Białynicki-Birula and Rosenlicht [7], 1962.

Theorem 1.4 (Białynicki-Birula, Rosenlicht) — *Let k be an algebraically closed field of characteristic zero. Let $F : k^n \to k^n$ be a polynomial map. If F is injective, then F is surjective and the inverse is a polynomial map, i.e. F is a polynomial automorphism.*

So the Jacobian Conjecture is equivalent to: if $F'(z) \neq 0$ for all $z \in \mathbb{C}^n$, then F is injective or equivalently if $F(a) = F(b)$ for some $a \neq b$, $a, b \in \mathbb{C}^n$ then $F'(z) = 0$ for some $z \in \mathbb{C}^n$, which is exactly the 'Rolle form' of the Jacobian Conjecture described in the question above.

The Jacobian Conjecture was first formulated as a question by O. Keller in the case $n = 2$ for polynomials with integer coefficients ([35], 1939). Therefore the Jacobian Conjecture is also called Keller's problem by several authors. Over the years many people have tried to prove the Jacobian Conjecture. As a result many false proofs have been given and even several of them are published (for an account on these 'proofs' we refer to the paper [4]). But more importantly the study of the Jacobian Conjecture has given rise to several surprising results concerning polynomial automorphisms and many interesting relations with other problems.

In the remainder of this section we will describe the present status of the n-dimensional Jacobian Conjecture.

So from now on let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Put

$$\deg(F) := \max_i \deg(F_i)$$

where $\deg(F_i)$ means the total degree of F_i.

From linear algebra we know that the Jacobian Conjecture is true if $\deg(F) = 1$. So the next case is $\deg(F) = 2$. It was only in 1980 that Stuart Wang proved that in this case the Jacobian Conjecture is true:

Proposition 1.5 (Wang, [58]) — *If $\deg(F) \leq 2$, then the Jacobian Conjecture is true.*

Proof. By theorem 1.4 it suffices to prove that F is injective. So suppose $F(a) = F(b)$ for some $a, b \in \mathbb{C}^n$, $a \neq b$. We first show that we can assume that $b = 0$. To see this we define $G(X) := F(X + a) - F(a)$. Then $\deg(G) \leq 2$, $G(0) = 0$ and putting $c := b - a$ we have $c \neq 0$ and $G(c) = 0$. Observe $JG(X) = (JF)(X + a)$, so $\det(JG) \in \mathbb{C}^*$. Now write $G = G_{(1)} + G_{(2)}$, its decomposition in homogeneous components. Consider $G(tc) = tG_{(1)}(c) + t^2 G_{(2)}(c)$. Differentiation gives

$$G_{(1)}(c) + 2tG_{(2)}(c) = \frac{d}{dt} G(tc) = JG(tc) \cdot c \neq 0$$

for all $t \in \mathbb{C}$, since $c \neq 0$ and $\det(JG) \in \mathbb{C}^*$. Substituting $t = \frac{1}{2}$ gives $G(c) \neq 0$, a contradiction with $G(c) = 0$. So F is injective. $\square$

Now one could think that this result is just a small improvement of the case $\deg(F) = 1$. However we have

Theorem 1.6 (Bass, Connell, Wright, [4], Yagzhev, [61]) — *If the Jacobian Conjecture holds for all $n \geq 2$ and all F with $\deg(F) \leq 3$, then the Jacobian Conjecture holds.*

In fact they even proved that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all F of the form

$$F = (X_1 + H_1, \ldots, X_n + H_n) \tag{1}$$

where each H_i is either zero or homogeneous of degree 3.

A little later this result was improved by Drużkowski:

Theorem 1.7 (Drużkowski, [17]) — *If the Jacobian Conjecture holds for all $n \geq 2$ and all F of the form*

$$F = \left(X_1 + \left(\sum a_{j1} X_j\right)^3, \ldots, X_n + \left(\sum a_{jn} X_j\right)^3\right) \tag{2}$$

then the Jacobian Conjecture holds.

What is known about the Jacobian Conjecture for the maps of the form (1) resp. (2)?

In 1993 David Wright in [60] showed that in case $n = 3$ the Jacobian Conjecture holds for all F of the form (1). In that paper Wright writes:

> 'Here it becomes useful to assume F is cubic homogeneous, since this limits the number of its monomials. The dimension four case may still be out of range even with this reduction, however; the number of monomials of degree three in four variables is 20, so the number of monomials for a cubic homogeneous map in dimension four is $20 \times 4 = 80$.'

Nevertheless Engelbert Hubbers (University of Nijmegen) succeeded in 1994 to solve the large system of polynomial equations (induced by $\det(JF) = 1$) with the help of a strong computer. So he showed that in case $n = 4$ the Jacobian Conjecture holds for all F of the form (1). In fact he completely classified all maps of the form (1) satisfying $\det(JF) = 1$. His main result is

Theorem 1.8 (Hubbers, [31]) — *Let $F = X - H$ be a cubic homogeneous polynomial map in dimension four, such that $\det(JF) = 1$. Then there exists some $T \in GL_4(\mathbb{C})$ with $T^{-1} \circ F \circ T$ being one of the following forms:*

1. $$\begin{pmatrix} x_1 & \\ x_2 & \\ x_3 & \\ x_4 & -a_4 x_1^3 - b_4 x_1^2 x_2 - c_4 x_1^2 x_3 - e_4 x_1 x_2^2 - f_4 x_1 x_2 x_3 \\ & -h_4 x_1 x_3^2 - k_4 x_2^3 - l_4 x_2^2 x_3 - n_4 x_2 x_3^2 - q_4 x_3^3 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 & \\ x_4 & -x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix}$$

3. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2 - r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ & +\frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 & \\ x_4 & -x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$

4. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 & -e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix}$$

5. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 & -x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ & -s_3x_3x_4^2 - t_3x_4^3 \\ x_4 & \end{pmatrix}$$

6. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ & -p_3x_2x_4^2 - t_3x_4^3 \\ x_4 & \end{pmatrix}$$

7. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 & -x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ & -n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$