Harrison Cocycles and the Group of Galois Coobjects

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Abstract

Let H be a commutative faithfully flat Hopf algebra over a commutative ring R. We give an exact sequence describing the group of H-Galois coobjects. The other terms in the sequence are Harrison cohomology groups. This generalizes an exact sequence due to Early and Kreimer and Yokogawa.

Résumé

Soit H une algèbre de Hopf commutative fidèlement plate sur un anneau commutatif R. Nous étudions une suite exacte qui décrit le groupe des co-objets H-Galois. Les autres termes de la suite sont des groupes de cohomologie de Harrison. Cela généralise une suite exacte due à Early, Kreimer et Yukogawa.

Introduction

Let H be a finite (i.e. a finitely generated projective) cocommutative Hopf algebra over a commutative ring R. Chase and Sweedler [4] introduced the notion of H-Galois object, generalizing classical Galois theory. Isomorphism classes of H-Galois objects form a group $\mathrm{Gal}(R, H)$. The multiplication on $\mathrm{Gal}(R, H)$ is induced by the cotensor product $\square_H$. Early and Kreimer [5] and, independently, Yokogawa [13] showed that $\mathrm{Gal}(R, H)$ fits into an exact sequence

$$(1) \qquad 1 \longrightarrow H^2(H, R, \mathbb{G}_m) \xrightarrow{\alpha} \mathrm{Gal}(R, H) \xrightarrow{\beta} H^1(H, R, \mathrm{Pic}) \xrightarrow{\gamma} H^3(H, R, \mathbb{G}_m)$$

Here the cohomology groups are Sweedler cohomology groups, cf. [11]. The definition of a Galois object can be generalized to the situation where H is not necessarily finitely generated or projective ([9]). The idea is the following: consider an H-comodule algebra A. Then we have a pair of adjoint functors between the category R-mod and the category of relative (A, H)-modules. This category consists of

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R-modules equipped with an A-action and an H-coaction satisfying a certain compatibility relation. If H is finite, then relative Hopf modules correspond to (right) $A^{\mathrm{opp}}\#H^*$-modules, and this explains the relation with the theory of Chase and Sweedler. If the adjunction is a category equivalence, then we say that A is an H-Galois object.
The question that we are interested in is the following: can we generalize the exact sequence (1) to the situation where the Hopf algebra H is not necessarily finitely generated and projective? The proofs exhibited in [5] and [13] make intensive use of the fact that the Hopf algebra H (and the H-Galois objects) are finitely generated and projective. This allows to switch back and forth between H-comodule algebras and H^*-module coalgebras. For example, the map β is given by forgetting the algebra structure, followed by taking the dual. We then obtain an H^*-module, representing a Sweedler cocycle. Of course these duality arguments no longer hold in the case where H is infinite. Another problem is the fact that the cotensor product is not naturally associative (unless we work over a field instead of a commutative ring). Moreover, we cannot prove that the cotensor product of two H-Galois objects is again an H-Galois object.
In this note, we propose to work with H-module coalgebras instead of H-comodule algebras. In [9], Schneider introduces a Galois theory for H-module coalgebras, leading to the notion of H-Galois coobject. If H is finite, then the dual of an H-Galois coobject is an H^*-Galois object. We will show that, for H commutative, the set of isomorphism classes of H-Galois coobjects forms a group $\mathrm{Gal}^{\mathrm{co}}(R,H)$. The operation is now induced by the tensor product $\otimes_H$. $\mathrm{Gal}^{\mathrm{co}}(R,H)$ fits into an exact sequence, and, in the case where H is finite, a duality argument shows that the exact sequence (1) follows from this new sequence.
When we try to add the H^3-term to the sequence, we face a phenomenon that is typical for the infinite case. We have to restrict attention to a subgroup of the group of Galois coobjects. This subgroup is defined as follows: consider Galois coobjects that have normal basis after we take a faithfully flat base extension. We will say that such a Galois coobject has a *geometric normal basis*. Thus a Galois coobject C has a geometric normal basis if $C \otimes S \cong H \otimes S$ as $H \otimes S$-modules for some faithfully flat commutative R-algebra S. If H is finite then all Galois coobjects have a geometric normal basis, we can take a Zariski covering for S. We have to apply a similar construction for the Picard group, and then we can state the generalized exact sequence, see Theorem 3.4.
Along the way, we obtain two results that seem to be new even in the finite case: we have an explicit construction for the inverse of an H-Galois coobject (Theorem 2.2), and, conversely, if an H-module coalgebra is a twisted form of H as an H-module and is invertible as an H-module coalgebra, then it is an H-Galois coobject

(Corollary 3.5).
Some additional difficulties arise if we try to construct a similar theory for Galois objects; moreover, the formalism turns out to be much more complicated in this situation, and this is why the author has the opnion that the coalgebra formalism is the natural formalism for this type of problem.
For standard results and terminology about Hopf algebras, we refer to the literature, e.g. [1], [7] or [11]. The reader should keep in mind that we work here over a commutative ring, while the monographs cited above restrict attention to Hopf algebras over a field.

1 Notations and preliminary results

Throughout this paper, H will be a commutative Hopf algebra over a commutative ring R, and assume that H is faithfully flat as an R-module. For the comultiplication on H we will use Sweedler's sigma notation ([10]):

$$\Delta(h) = \sum h_1 \otimes h_2$$

A left H-*module coalgebra* is an R-module C such that C is a left H-module and an R-coalgebra satisfying the compatibility relations

$$\Delta_C(h \rightharpoonup c) = \sum (h_1 \rightharpoonup c_1) \otimes (h_2 \rightharpoonup c_2) \tag{2}$$

$$\varepsilon_C(h \rightharpoonup c) = \varepsilon_H(h)\varepsilon_C(c) \tag{3}$$

for all $h \in H$ and $c \in C$. The left action of H on C is denoted by $\rightharpoonup$. If H is commutative, it makes no sense to distinguish between left and right H-module coalgebras.
Let C be a left H-module coalgebra. Then a left (H, C)-Hopf module M is an R-module that is a left H-module and a left C-comodule such that

$$\rho_M(h \cdot m) = \sum h_1 \rightharpoonup m_{(-1)} \otimes h_2 m_{(0)} \tag{4}$$

for all $m \in M$ and $h \in H$. In the sequel, ${}_H^C\mathcal{M}(H)$ will denote the category of left (H, C)-Hopf modules and H-linear C-colinear maps.

Proposition 1.1 — *With notations as above, consider the functors*

$$F : {}_H^C\mathcal{M}(H) \longrightarrow R\text{-mod} : M \mapsto R \otimes_H M = \overline{M}$$
$$G : R\text{-mod} \longrightarrow {}_H^C\mathcal{M}(H) : N \mapsto C \otimes N$$

Then G is a right adjoint to F.

Proof. This result is a special case of [3, Theorem 1.3]. We restrict to giving a brief sketch of the proof. R is an H-module via the map ε. In fact, $\overline{M} = M/\mathrm{Ker}\varepsilon M$, and in $\overline{M}$ we have the following identity:

$$\overline{hm} = \varepsilon(h)\overline{m}$$

for all $h \in H$ and $m \in M$. For any $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod we consider the maps

$$\alpha : \mathrm{Hom}_H^C(M, C \otimes N) \longrightarrow \mathrm{Hom}_R(\overline{M}, N)$$
$$\beta : \mathrm{Hom}_R(\overline{M}, N) \longrightarrow \mathrm{Hom}_H^C(M, C \otimes N)$$

given by

$$\alpha(f)(\overline{m}) = (\varepsilon_C \otimes I_N)(f(m))$$
$$\beta(g)(m) = \sum m_{(-1)} \otimes g(\overline{m}_{(0)})$$

for all $f \in \mathrm{Hom}_{{}^C_H\mathcal{M}(H)}(M, C \otimes N)$, $g \in \mathrm{Hom}_R(\overline{M}, N)$ and $m \in M$. A straightforward verification shows that f and g are well-defined and each others inverses. This finishes the proof. □

From the adjointness of the functors F and G in Proposition 1.1, it follows that for all $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod we have natural maps

$$\psi_M : M \longrightarrow G(F(M)) = C \otimes \overline{M}$$
$$\phi_N : F(G(N)) = \overline{C} \otimes N \longrightarrow N$$

given by

$$\psi_M(m) = \sum m_{(-1)} \otimes \overline{m}_{(0)}$$
$$\phi_N(\sum_i \overline{c}_i \otimes n_i) = \sum_i \varepsilon(c_i) n_i$$

Definition 1.2 — *With notations as above, an H-module coalgebra C is called an H-Galois coobject if the functors F and G from Proposition 1.1 are inverse equivalences, or, equivalently, if ψ_M and ϕ_N are isomorphisms for all $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod.*

We will now establish some necessary and sufficient conditions for an H-module coalgebra to be an H-Galois coobject. It is clear that ϕ_N is an isomorphism for all $N \in R$-mod if and only if the canonical map

$$\phi_C : \overline{C} \longrightarrow R : \overline{c} \mapsto \varepsilon(c)$$

is an isomorphism.
Observe that $H \otimes C$ can be given the structure of left (H, C)-Hopf module as follows:

$$k(h \otimes c) = kh \otimes c$$
$$\rho_{H\otimes C}(h \otimes c) = \sum h_1 \rightharpoonup c_1 \otimes h_2 \otimes c_2$$

for all $h, k \in H$ and $c \in C$. It is readily verified that condition 4 is satisfied:

$$\begin{aligned} \rho_{H\otimes C}(kh \otimes c) &= \sum k_1 h_1 \rightharpoonup c_1 \otimes k_2 h_2 \otimes c_2 \\ &= \sum k_1 (h \otimes c)_{(-1)} \otimes k_2 (h \otimes c)_{(0)} \end{aligned}$$

A necessary condition for M to be an H-Galois coobject is therefore that $\delta = \psi_{H\otimes C}$ is an isomorphism. Let us describe δ. First we remark that $F(H \otimes C) = \overline{H} \otimes C = C$, since $\overline{H} \cong R$. Indeed, the maps

$$I \otimes \varepsilon_H : \ \overline{H} = R \otimes_H H \longrightarrow R \qquad \eta \otimes 1 : \ R \longrightarrow \overline{H} = R \otimes_H H$$

are well-defined and each others inverses.
Now $G(F(H \otimes C)) = C \otimes C$, where H acts and C coacts on the first factor:

$$h(c \otimes d) = h \rightharpoonup c \otimes d$$
$$\rho_{C\otimes C}(c \otimes d) = \sum c_1 \otimes c_2 \otimes d$$

$\delta = \psi_{H\otimes C}$ is given by the formula

$$\delta(h \otimes c) = \sum (h_1 \rightharpoonup c_1) \otimes \varepsilon(h_2) c_2 = \sum (h \rightharpoonup c_1) \otimes c_2$$

Theorem 1.3 **—** *Let H be a commutative, faithfully flat Hopf algebra. For a left H-module coalgebra C, the following conditions are equivalent:*

1. *C is an H-Galois coobject;*
2. – *$\overline{C} = R$;*
   – *$\delta = \psi_{H\otimes C} : \ H \otimes C \longrightarrow C \otimes C : \ h \otimes c \mapsto \sum (h \rightharpoonup c_1) \otimes c_2$ is an isomorphism;*
   – *C is flat as an R-module.*
3. – *$\delta = \psi_{H\otimes C} : \ H \otimes C \longrightarrow C \otimes C : \ h \otimes c \mapsto \sum (h \rightharpoonup c_1) \otimes c_2$ is an isomorphism;*
   – *C is faithfully flat as an R-module.*

Proof. For full detail, we refer to [9] or to [3], where more general results are given. The reader might object that the results in [3] are valid only if we work over a field k, but it can be verified that the above Theorem is true over a commutative ring. ☐

Corollary 1.4 **—** *Let H be a commutative, faithfully flat Hopf algebra. Then H viewed as a left H-module coalgebra is an H-Galois coobject.*