Interrelations between Mathematics and Physics

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Abstract

After briefly describing the mathematical structure of modern physics, this paper analyzes the divergence between the development of physics and of mathematics in the first half of the 20th century, with emphasis on the role in each discipline of rigorous definitions and proofs, of algebraic calculations and of intuitive ideas.

Résumé

Après avoir décrit la structure mathématique de la physique moderne, cet article analyse la divergence entre mathématiques et physique dans la première moitié du XX^{e} siècle, en étudiant, pour chacune des disciplines, le rôle respectif des définitions et démonstrations rigoureuses, des calculs algébriques et des idées intuitives.

1. Foreword

I would like to start with an explicit description of the conceptual framework of this study.

To render it concisely, it is useful to look at the case of comparative linguistics. The history of a language is not a history of all, or even of "the most important," utterances (oral or written) in this language. Rather, it is a history of evolution *of the language as a system.* Hence we need a preliminary description of the system(s) whose genesis we are studying.

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An application of this Saussurian scheme to the history of mathematics (which, incidentally, I do not consider to be a mere language) was probably particularly appealing to Jean Dieudonné who, as an active member of the Bourbaki group, participated in the creation of a systematic picture of modern mathematics.[1] In this talk I follow his example, on a much humbler scale. Needless to say that restrictions of time, space, and competence, force me to choose a thin chain of connected ideas and present them in a highly selective way.

Thus I refuse (somewhat reluctantly) to discuss the history with Rankean insistence on *wie es eigentlich gewesen ist.* One reason for this refusal is that the history of contemporary mathematics tends to degenerate into credit and priority assignments, lacking pathetically the dramatic appeal with which the history of struggles for real power is charged. A more personal and compelling motive is succinctly put by Joseph Brodsky in his autobiographical essay *Less Than One*: "The little I remember becomes even more diminished by being recollected in English."

A last word of warning and apology is due. Any system is, of course, a theoretical construct. As such, it is at best relative and culture dependent, at worst subjective. It is precisely in this function that it can serve as material for the history of mathematics of the 20th century.

2. Mathematical Physics as a System

2.1. Physics

Physics describes the external world, and in its domain of competence, does this in two complementary modes: classical and quantum.

In the *classical mode,* events occur to the matter and fields which reside and evolve in the space–time. Physical laws directly constrain observables. They are basically deterministic and expressed by the differential equations which (sometimes demonstrably, sometimes hypothetically) satisfy appropriate uniqueness and existence theorems.

A statistical submode of the classical mode of description deals with probabilities and averages which (sometimes demonstrably, sometimes presumably) can be deduced from an ideal deterministic description. The need for a statis-

[1] Jean Dieudonné, as I remember him, had a strong voice, strong hands, and strong opinions. In particular, he insisted on using tensor products and commutative diagrams instead of classical subscripts and superscripts in calculations involving tensors. I used to believe his judgement that this was a chalk–saving device, until one day I had to calculate with tensors myself. Then I found out that subscripts were much more economical.

tical treatment arises from two basic premises: too many degrees of freedom and/or instability. (Metaphorically speaking, instability means that each consecutive decimal digit is a new degree of freedom.)

A fundamental physical abstraction is that of *an isolated system* which evolves in oblivion of the rest of the world, and of *interaction* between potentially isolated systems, or one isolated system and the rest of the world.

In one of the most remarkable flights of fancy of classical physics, *space–time* itself appears as such an isolated system governed by Einstein's equations of general relativity (perhaps, with an energy–momentum tensor summarily responsible for everything which is not pure space–time).

In the *quantum mode* of theoretical description, the observable world is inherently probabilistic. Moreover, and more significantly, the basic laws — which are in a sense deterministic — govern an unobservable entity, the *probability amplitude*, which is a complex valued function on a quantum path space. Roughly speaking, the amplitude of a composite event is the product of the amplitudes of its constituents, whereas the amplitude of an event which is a sum of alternatives is the sum of the amplitudes of these alternatives.

The probability of an event is the modulus squared of its amplitude. Physical observables are the appropriate averages, even if one speaks about an elementary act of scattering of an individual particle. The observable wave behavior of, say, light is only an imperfect reflection of the inherent wave behavior of the amplitudes (wave functions) of an indeterminate number of photons described by the Fock space of the quantized electromagnetic field.

Partly as a result of historical development, many quantum models contain as an intermediate stage a classical model which is then quantized. The word "quantization" rather indiscriminately refers to a wide variety of procedures of which two of the most important are operator, or Hamiltonian, quantization, and *path integral* quantization. The first is more algebraic and usually has a firmer mathematical background. The second possesses an enormous heuristic and aesthetic potential. I haven chosen the latter for my more detailed subsequent discussion.

If I had included the first one, the picture of the divergence of Mathematics and Physics in the first half of this century sketched below in Sec. IV would appear less pronounced. Nevertheless, the main results of my analysis would survive.

One more subject matter deserving a separate historical and structural study is the duality between these two approaches. It started with classical mechanics, Lagrange, and Hamilton, and continued via Heisenberg–Schrödinger wave mechanics to the path integral/scattering matrix controversy. On the fringes of physics it contains such recent mathematical gems as

Virasoro algebra representations on the moduli spaces of curves.

2.2. Mathematics

If there is one most important notion of mathematical physics, it is that of *action* functional. It encompasses the classical ideas of energy and work, its density in a domain of space–time is the Lagrangian, and multiplied by $\sqrt{-1}$ and exponentiated, it furnishes the basic probability amplitude. Action is measured in absolute Planck units, and therefore can be thought of as a real number. More precisely, we will consider the following scheme of description central for both modes of physical description referred to above.

The modeling of a physical system starts with the specification of its kinematics. This includes a space $\mathcal{P}$ of virtual classical paths of the system and an action functional $S: \mathcal{P} \to \mathbb{R}$. For example, $\mathcal{P}$ may consist of parametrized curves in a classical phase space of a mechanical system, or of Riemannian metrics on a given smooth manifold (space–time), or of triples *(a connection on a given vector bundle, a metric on it, a section of it)* etc. The value of the action functional at a point $p \in \mathcal{P}$ is usually given in the form $\int_p L$, that is a volume form integrated over one of the spaces figuring in the description of p.

Classical equations of motion specify a subspace $\mathcal{P}_{cl} \subset \mathcal{P}$. This subset consists of the solutions of the variational equations $\delta(S) = 0$, *i. e.,* of the stationary points of the action functional.

If the classical description is the statistical one, then $\exp(-S)$ is the probability density.

In the quantum description, we choose physically motivated subsets $B \subset \mathcal{P}$, typically determined by boundary conditions, and define the average of an observable O in B by a path integral of the type

$$\langle O\rangle_B := \int_B O(p)\, e^{i\int_p L}\, Dp. \tag{2.1}$$

These are our main actors. In the following, I present some musings about the history of this picture as seen through the eyes of physicists and mathematicians.

I will be most interested in the idea of the integral and its final incarnation, in the form of the *path integral.*

3. The Integral

The notion of an integral is one of the central and recurring themes in the history of mathematics for the last two millennia. The ardent problem solving is periodically followed by the anxious definition seeking, only to be replaced by new non–rigorous but amazingly efficient heuristics leaving a logically–minded fundamentalist in each of us baffled.

Richard Feynman who created the hierogram (2.1) (still lacking a precise mathematical meaning exactly in those cases when it is most needed by physicists[2]) used to boast that (2.1) allowed the calculation of the anomalous magnetic momentum of the electron, which coincided with its experimental value up to ten digits:

"As of 1983, the theoretical number was 1.00115965246, with an uncertainty of about 20 in the last two digits; the experimental number was 1.00115965221, with an uncertainty of about 4 in the last digit. This accuracy is equivalent to measuring the distance from Los Angeles to New York, a distance of over 3000 miles, to within the width of a human hair." [Feynman 1988, p. 118]

This feat was recently matched by physical calculations (even called "predictions", cf. [Candelas *et al.* 1991]) of various interesting numbers in algebraic geometry, such as the number N_d of rational curves of degree d on a generic three–dimensional quintic (e. g. 70428 81649 78454 68611 34882 49750 for $d = 10$, a theoretical(?) number still unchecked in an experiment(?) involving a mathematical definition of N_d and a computer.) The ideology of path integration played an essential role in these calculations, leading to an interpretation of an instance of (2.1) as a sum over instantons in a sigma–model, which in this particular case are rational curves on a quintic.

The intuitive physical picture of an integral is *the quantity of something in a domain.* If the first calculations of this "something" are later interpreted as, say, the volume of a pyramid, one can hardly doubt that they were used for estimating the actual quantity of *stone* (and slaves' labor) needed for the building of an Egyptian pharaoh's tomb. Kepler's *Stereometria Doliorum* mentions *wine casks* in its title. The domain in question acquired a temporal dimension when *the length* of a path was calculated as an integral of velocity, and the notion of energy was gradually replaced by that of *action.* In the twentieth century, *topology* became one of the substances the quantity of which could be measured by integration of closed differential forms (De Rham theory of periods anticipated by Poincaré). *Probability* turned out to be another such substance, and Wiener's treatment of Brownian motion as a measure in a space of continuous paths paved the way both for Kolmogorov's axiomatic

[2]For a more positive view, see [Glimm and Jaffe 1981], a remarkable book which influenced the structure of this essay. On page 313 however the authors say: "... it is a theoretical puzzle whether a *theory* of electrodynamics exists in the sense of a mathematical framework ..."