From Riemann Surfaces to Complex Spaces

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We must always have old memories and young hopes

Abstract

This paper analyzes the development of the theory of Riemann surfaces and complex spaces, with emphasis on the work of Riemann, Klein and Poincaré in the nineteenth century and on the work of Behnke-Stein and Cartan-Serre in the middle of this century.

Résumé

Cet article analyse le développement de la théorie des surfaces de Riemann et des espaces analytiques complexes, en étudiant notamment les travaux de Riemann, Klein et Poincaré au XIX^{e} siècle et ceux de Behnke-Stein et Cartan-Serre au milieu de ce siècle.

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This exposé is an enlarged version of my lecture given in Nice. *Gratias ago* to J.-P. Serre for critical comments. A detailed exposition of sections 1 and 2 will appear elsewhere.

1. Riemann surfaces from 1851 to 1912

1.1. Georg Friedrich Bernhard Riemann and the covering principle

The *theory of Riemann surfaces* came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of analysis, topology and algebra, she sprang forth from Riemann's Jovian head (cf. H. Weyl, [*Ges. Abh.* III, p. 670]). Indeed on November 14, 1851, Riemann submitted a thesis *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (Foundations of a general theory of functions of one complex variable) to the faculty of philosophy of the University of Göttingen to earn the degree of doctor philosophiae. Richard Dedekind states in "Bernhard Riemann's Lebenslauf", that Riemann had probably conceived the decisive ideas in the autumn holidays of 1847, [Dedekind 1876, p. 544]. Here is Riemann's definition of his surfaces as given in [Riemann 1851, p. 7]:

"Wir beschränken die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes O nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. ... Wir lassen die Möglichkeit offen, dass der Ort des Punktes O über denselben Theil der Ebene sich mehrfach erstrecke, setzen jedoch für einen solchen Fall voraus, dass die auf einander liegenden Flächentheile nicht längs einer Linie zusammenhängen, so dass eine Umfaltung der Fläche, oder eine Spaltung in auf einander liegende Theile nicht vorkommt."

(We restrict the variables x, y to a finite domain by considering as the locus of the point O no longer the plane A itself but a surface T spread over the plane. We admit the possibility ... that the locus of the point O is covering the same part of the plane several times. However in such a case we assume that those parts of the surface lying on top of one another are not connected along a line. Thus a fold or a splitting of parts of the surface cannot occur).

Here the plane A is the complex plane $\mathbb{C}$, which Riemann introduces on page 5. Later, on page 39, he also admits "die ganze unendliche Ebene A", *i.e.,* the sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. It is not clear what is meant by "mehrfach erstrecke". Does he allow only finitely or also infinitely many points over a point of A? The last lines in Riemann's definition are vague: his intention is to describe local branching *topologically.* For algebraic functions this had already been done in an *analytic* manner by V. Puiseux [1850]. A careful discussion of the notion of "Windungspunkt $(m-1)$ Ordnung" (winding point of order $m-1$) is given by Riemann on page 8.

Riemann's definition is based on the *covering principle*: let $z : T \to \hat{\mathbb{C}}$ be a continuous map of a topological surface T into $\hat{\mathbb{C}}$. Then T is called a *(concrete) Riemann surface* over $\hat{\mathbb{C}}$ (with respect to z) if the map z is *locally finite*[1] and a local homeomorphism outside of a locally finite subset S of T. In this case there exists around every point $x \in X$ a local coordinate t with

[1] This means that to every point $x \in T$ there exist open neighborhoods U, resp. V, of x, resp. $z(x)$, such that z induces a finite map $U \to V$.

$t(x) = 0$. If $z(x) = z_0$, resp. $z(x) = \infty$, the map z is given by $z - z_0 = t^m$, resp. $z = t^{-m}$, with $m \in N \setminus \{0\}$ and $m = 1$ whenever $x \notin S$. A unique complex structure (cf. section 1.2.) on T such that $z : T \to \hat{\mathbb{C}}$ is a meromorphic function is obtained by lifting the structure from $\hat{\mathbb{C}}$; the winding points are contained in S.

The requirements for the map $z : T \to \hat{\mathbb{C}}$ can be weakened. According to Simion Stoilow it suffices to assume that z is continuous and open and that no z-fiber contains a continuum [Stoilow 1938, chap. V].

Riemann's thesis is merely the sketch of a vast programme. He gives no examples, *Aquila non captat muscas* (Eagles don't catch flies). The breath-taking generality was at first a hindrance for future developments. Contrary to the *Zeitgeist*, holomorphic functions are defined by the Cauchy-Riemann differential equations. Explicit representations by power series or integrals are of no interest. Formulae are powerful but blind. On page 40 Riemann states his famous mapping theorem. His proof is based on Dirichlet's principle.

Six years later, in his masterpiece "Theorie der Abel'schen Funktionen", Riemann [1857] explains the intricate connections between algebraic functions and their integrals on compact surfaces from a bird's-eye view of (not yet existing) *analysis situs*. The number p, derived topologically from the number $2p + 1$ of connectivity and called *"Geschlecht" (genus)* by Clebsch in [Clebsch 1865, p. 43], makes its appearance on p.104 and "radiates like wild yeast through all meditations". The famous inequality $d \geq m - p + 1$ for the dimension of the $\mathbb{C}$-vector space of meromorphic functions having at most poles of first order at m given points occurs on pages 107-108; Gustav Roch's refinement in [Roch 1865] became the immortal *Riemann-Roch theorem*. The equation $w = 2n + 2p - 2$ connecting genus and branching, which was later generalized by Hurwitz to the *Riemann-Hurwitz formula*, [Hurwitz 1891, p. 376; 1893, pp. 392 and 404], is derived by analytic means on page 114.

Riemann and many other great men share the fate that at their time there was no appropriate language to give their bold way of thinking a concise form. In 1894 Felix Klein wrote, [1894, p. 490]: "Die Riemannschen Methoden waren damals noch eine Art Arcanum seiner direkten Schüler und wurden von den übrigen Mathematikern fast mit Mißtrauen betrachtet" (Riemann's methods were kind of a secret method for his students and were regarded almost with distrust by other mathematicians). M. A. Stern, Riemann's teacher of calculus in Göttingen, once said to F. Klein [1926, p. 249]: "Riemann sang damals schon wie ein Kanarienvogel" (Already at that time Riemann sang like a canary).

Poincaré wrote to Klein on March 30, 1882: "C'était un de ces génies qui renouvellent si bien la face de la Science qu'ils impriment leur cachet, non seulement sur les œuvres de leurs élèves immédiats, mais sur celles de tous leurs successeurs pendant une longue suite d'années. Riemann a créé une théorie nouvelle des fonctions" [Poincaré 1882b, p. 107]. Indeed "Riemann's writings are full of almost cryptic messages to the future. ⋯ The spirit of Riemann will move future generations as it has moved us" [Ahlfors 1953, pp. 493, 501].

1.1*. Riemann's doctorate With his request of November 14, 1851, for admission to a doctorate, Riemann submits his vita. Of course this is in Latin as the university laws demanded. On the following day, the Dean informs the faculty:

> It is my duty to present to my distinguished colleagues the work of a new candidate for our doctorate, Mr. B. Riemann from Breselenz; and entreat Mr. Privy Councillor Gauss for an opinion on the latter and, if it proves to be satisfactory, for an appropriate indication of the day and the hour when the oral examination could be held. The candidate wants to be examined in mathematics and physics. The Latin in the request and the vita is clumsy and scarcely endurable: however, outside the philological sciences, one can hardly expect at present anything better, even from those who like this candidate are striving for a career at the university.
>
> 15 Nov., 51. Respectfully,
>
> Ewald

Gauss complies with the Dean's request shortly thereafter (undated, but certainly still in November 1851); the great man writes in pre-Sütterlin calligraphy the following "referee's report":

> The paper submitted by Mr. Riemann bears conclusive evidence of the profound and penetrating studies of the author in the area to which the topic dealt with belongs, of a diligent, genuinely mathematical spirit of research, and of a laudable and productive independence. The work is concise and, in part, even elegant: yet the majority of readers might well wish in some parts a still greater transparency of presentation. The whole is a worthy and valuable work, not only meeting the requisite standards which are commonly expected from doctoral dissertations, but surpassing them by far.
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> I shall take on the examination in mathematics. Among weekdays Saturday or Friday or, if need be, also Wednesday is most convenient to me and, if a time in the afternoon is chosen, at 5 or 5:30 p.m. But I also would have nothing to say against the forenoon hour 11 a.m. I am, incidentally, assuming that the examination will not be held before next week.
>
> Gauss