On the History of Hilbert's Twelfth Problem

A Comedy of Errors

Norbert Schappacher*

Abstract

Hilbert's 12th problem conjectures that one might be able to generate all abelian extensions of a given algebraic number field in a way that would generalize the so-called theorem of Kronecker and Weber (all abelian extensions of $\mathbb{Q}$ can be generated by roots of unity) and the extensions of imaginary quadratic fields (which may be generated from values of modular and elliptic functions related to elliptic curves with complex multiplication). The first part of the lecture is devoted to the false conjecture that Hilbert made for imaginary quadratic fields. This is discussed both from a historical point of view (in that Hilbert's authority prevented this error from being corrected for 14 years) and in mathematical terms, analyzing the algebro-geometric interpretations of the different statements and their respective traditions. After this, higher-dimensional analogues are discussed. Recent developments in this field (motives, etc., also Heegner points) are mentioned at the end.

Résumé

Le douzième problème de Hilbert propose une façon conjecturale d'engendrer les extensions abéliennes d'un corps de nombres, en généralisant le théorème dit de Kronecker et Weber (toutes les extensions abéliennes de $\mathbb{Q}$ sont engendrées par des racines de

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*Université Louis Pasteur, I.R.M.A., 7 rue René Descartes, 67084 Strasbourg Cedex.

l'unité) ainsi que les extensions des corps quadratiques imaginaires (qui sont engendrées par des valeurs de fonctions modulaires et elliptiques liées aux courbes elliptiques à multiplication complexe). La première partie de l'exposé est centrée autour de la conjecture incorrecte de Hilbert dans le cas du corps quadratique imaginaire. Elle est dicutée aussi bien du point de vue historique (pendant quatorze ans, l'autorité de Hilbert empêcha la découverte de cette erreur), que du point de vue mathématique, en analysant les interprétations algébro-géometriques des énoncés différents relatifs à ce cas et de leurs traditions. On discute ensuite des analogues en dimension supérieure. Les développements récents (motifs, etc., aussi points de Heegner) sont mentionnés à la fin.

A good problem should be

- well motivated by already established theories or results,
- challenging by its scope and difficulty,
- sufficiently open or vague, to be able to fuel creative research for a long time to come, maybe for a whole century.

David Hilbert tried to follow these precepts in his celebrated lecture *Mathematische Probleme* at the Paris International Congress of Mathematicians in 1900.[1] He did not have time to actually present in his speech all 23 problems which appear in the published texts.[2] In particular, the 12th problem on the generalization of the Kronecker-Weber Theorem by the theory of Complex Multiplication did not make it into the talk. This may be due to the slight technicality of the statements involved. But Hilbert held this 12th problem in very high esteem. In fact, according to Olga Taussky's recollection, when he introduced Fueter's lecture "Idealtheorie und Funktionentheorie" at the 1932 International Congress at Zürich, Hilbert said that "the theory of complex multiplication (of elliptic modular functions) which forms a powerful link between number theory and analysis, is not only the most beautiful part of mathematics but also of all science."[3]

[1] [*ICM* 1900, pp. 58-114] (French translation by L. Laugel of an original German version), [Hilbert 1901] (definite German text), cf. [Alexandrov 1979].

[2] [Reid 1970, p. 81f]. See also *Enseign. Math.*, 2 (1900), pp. 349-355.

[3] Obituary Notice for Hilbert in *Nature*, 152 (1943), p. 183. I am grateful to J. Milne for giving me this reference. In [*ICM* 1932, p. 37], one reads about Hilbert presiding over this first general talk of the Zürich congress: "Der Kongress ehrt ihn, indem die Anwesenden sich von ihren Sitzen erheben."

The present article covers in detail a period where a number of initial mistakes by most mathematicians working on the problem were finally straightened out. At the end of the 1920's the explicit class field theory of imaginary quadratic fields was established and understood essentially the way we still see it today. However, the higher dimensional theory of singular values of Hilbert modular forms remained obscure. Later developments are briefly indicated in the final section of the paper.

What I describe here in detail *is* a comedy for us who look back. It is genuinely amusing to see quite a distinguished list of mathematicians pepper their contributions to Hilbert's research programmme with mistakes of all sorts, thus delaying considerably the destruction of Hilbert's original conjecture which happened to be not quite right. The comedy is at the same time a lesson on how, also in mathematics, personal authority influences the way research progresses — or is slowed down. It concerns the condition of the small group of researchers who worked on Hilbert's 12th problem. The errors made are either careless slips or delusions brought about by wishful thinking which was apparently guided by Hilbert's claim. The authors were just not careful enough when they set up a formalism which they controlled quite well in principle (a weakness in the formalism may, however, be behind the big error in Weber's false proof of the "Kronecker–Weber Theorem" — see section 2 below). Meanwhile Hilbert was conspicuously absent from the scene after 1900.[4] This is also not atypical for the comedy where the characters are mostly left to themselves when it comes to sorting out their complicated situation:

> "— Say, is your tardy master now at hand? ...
>
> — Ay, Ay, he told his mind upon mine ear.
>
> Beshrew his hand, I scarce could understand it.
>
> — Spake he so doubtfully, thou couldst not feel his meaning?
>
> — Nay, he struck so plainly, I could too well feel his blows; and withal so doubtfully, that I could scarce understand them."
>
> (Shakespeare, *The comedy of errors*, II-1)

The history of complex multiplication has already received a certain attention in the literature — see in particular the well-researched book [Vlăduţ 1991]. Apart from newly introducing a few details into the story, my main difference

[4]Hilbert did intervene indirectly, as thesis advisor. As such he should have been better placed han anybody else to see, for example, that Takagi's thesis of 1901 produced extensions that provided counterexamples to Fueter's thesis of 1903... See section 3 below.

with existing publications is the emphasis that I put on Hilbert's peculiar perspective of his problem, which is not only very much different from our current viewpoint, but seems also to be the very reason which led him to the slightly wrong conjecture for imaginary-quadratic base fields in the first place.

As for the style of exposition, I try to blend a general text which carries the overall story, with some more mathematical passages that should be understandable to any reader who knows the theories involved in their modern presentation.

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1. Hilbert's statement of the Twelfth Problem

Coming back to the features of a good problem stated at the beginning, let us look at the motivation which Hilbert chose for his 12th problem. He quoted two results.

First, a statement "going back to Kronecker," as Hilbert says, and which is known today as the "Theorem of Kronecker and Weber." It says that every Galois extension of $\mathbb{Q}$ with abelian Galois group is contained in a suitable cyclotomic field, *i.e.*, a field obtained from $\mathbb{Q}$ by adjoining suitable roots of unity. This was indeed a theorem at the time of the Paris Congress—although not proved by the person Hilbert quoted... We will briefly review the history of this result in section 2 below.

Second, passing to Abelian extensions of an imaginary quadratic field, Hilbert recalled the Theory of Complex Multiplication. As Hilbert puts it:

"Kronecker himself has made the assertion that the Abelian equations in the domain of an imaginary quadratic field are given by the transformation equations of the elliptic functions [*sic!*] with singular moduli so that, according to this, the elliptic function [*sic!*] takes on the role of the exponential function in the case considered before."[5] The slight incoherence of this sentence, which goes from certain "elliptic functions" (plural—as in Kro-

[5]"Kronecker selbst hat die Behauptung ausgesprochen, daß die Abelschen Gleichungen im Bereiche eines imaginären quadratischen Körpers durch die Transformationsgleichungen der elliptischen Funktionen mit singulären Moduln gegeben werden, so daß hiernach die elliptische Funktion die Rolle der Exponentialfunktion im vorigen Falle übernimmt." [Hilbert 1901, p. 311].

necker's[6] standard usage in this context) to "the elliptic function" (definite singular), is not a slip.[7] In fact, it gives the key to Hilbert's interpretation of Kronecker, and to his way of thinking of the 12th problem. What Hilbert actually means here becomes crystal clear in the final sentence on the 12th problem, because there he expands the singular "the elliptic function" into "the elliptic modular function."[8] So Hilbert was prepared, at least on this occasion, to use the term "elliptic function" also to refer to (elliptic) modular functions, *i.e.*, to (holomorphic, or meromorphic) functions $f : \mathcal{H} \longrightarrow \mathbb{C}$, where $\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ denotes the complex upper half plane, such that

$$f(\frac{a\tau+b}{c\tau+d}) = f(\tau), \qquad \text{for all } \ \tau \in \mathcal{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

And Hilbert's definite singular, "the elliptic (modular) function," refers undoubtedly to the distinguished holomorphic modular function $j : \mathcal{H} \longrightarrow \mathbb{C}$ which extends to a meromorphic function $j : \mathcal{H} \cup \{i\infty\} \longrightarrow \mathbb{C}$ with a simple pole at $i\infty$, where it is given (up to possible renormalization by some rational factor, in the case of some authors) by the well-known Fourier development in $q = e^{2\pi i \tau}$:

$$j(q) = \frac{1}{q} + 744q + 196884q + 21493760q^2 + \ldots$$

See for instance [Weber 1891, § 41] who calls this function simply "die Invariante," and cf. [Fueter 1905, p. 197], a publication on this problem which arose from a thesis under Hilbert's guidance.

To be sure, this was and is not at all the standard usage of the term "elliptic function." Rather, following Jacobi—despite original criticism from Legendre who had used the term to denote what we call today elliptic integrals—it was customary as of the middle of the 19th century to call elliptic functions the functions that result from the inversion of elliptic integrals, *i.e.*, the (meromorphic) doubly periodic functions with respect to some lattice. If one takes the lattice to be of the form $\mathbb{Z} + \mathbb{Z}\tau$, for $\tau \in \mathcal{H}$, then a typical example of such an elliptic function is Weierstrass's well-known $\wp$-function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}}{}' \Big(\frac{1}{(z-m\tau-n)^2} - \frac{1}{(m\tau+n)^2}\Big),$$

[6] For instance [Kronecker 1877, p. 70], [Kronecker 1880, p. 453]. Cf. section 4 below.

[7] Laugel missed this in his French translation of the text [*ICM* 1900, p. 88f], and thereby blurred the meaning of the sentence.

[8] "... diejenigen Funktionen ..., die für einen beliebigen algebraischen Zahlkörper die entsprechende Rolle spielen, wie die Exponentialfunktion für den Körper der rationalen Zahlen und die elliptische Modulfunktion für den imaginären quadratischen Zahlkörper." [Hilbert 1901, § 313].