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ON THE FIRST VAFA-WITTEN BOUND FOR TWO-DIMENSIONAL TORI

by

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Abstract. — In this paper we explicitly compute the first Vafa-Witten bound for a two-dimensional torus, namely the best uniform upper bound for the first eigenvalue of the family of twisted (by arbitrary vector potentials) Dirac operators on a flat two-torus. Starting with an arbitrary flat metric we give either an exact answer or a precise algorithm for producing an answer. As a by-product we develop a constructive way of implementing the projection map from the Poincaré upper half-plane onto the standard fundamental domain for its $SL(2, \mathbf{Z})$-action.

Résumé **(Sur la première borne de Vafa-Witten pour les tores de dimension deux)**
Dans cet article nous calculons explicitement la première borne de Vafa-Witten pour un tore de dimension 2, c'est-à-dire la meilleure borne supérieure pour la première valeur propre de la famille d'opérateurs de Dirac couplés à des potentiels vectoriels arbitraires, définis sur un tore plat de dimension 2. Pour une métrique plate arbitraire nous donnons soit la solution exacte de ce problème soit un algorithme précis pour en produire une. Une conséquence de nos résultats est une réalisation constructive de la projection du demi-plan de Poincaré sur le domaine fondamental de l'action de $SL(2, \mathbf{Z})$ sur celui-ci.

1. Introduction

Let M be a fixed compact Riemannian spin manifold with spinor bundle S and Dirac operator $\partial\!\!\!/$. For any Hermitian vector bundle E with metric connection A form the twisted Dirac operator $\partial\!\!\!/_A$ acting on $S \otimes E$. In a remarkable paper [**VW**], also [**A**], Vafa and Witten proved, among other things, that if the discrete eigenvalues of $\partial\!\!\!/_A$ are indexed by increasing absolute value,

$$|\lambda_1| \leq |\lambda_2| \leq \dots,$$

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then there is a bound C_1, which depends on M but not on the twisting data (E, A), such that

$$|\lambda_1| \leq C_1. \tag{1.1}$$

Subsequently, Moscovici [**M**] extended the inequality (1.1) to noncommutative geometric spaces, in the sense of Connes [**C**], which have finite topological type and satisfy rational Poincaré duality in K-theory.

Vafa and Witten [loc.cit.] also addressed the problem of finding the best bound C_1 in (1.1), if M is the d-dimensional torus $\mathbf{T}^d$ with angular variables $\phi^1, \phi^2, \ldots, \phi^d$, and flat metric $ds^2 = \sum_{i,j} g_{ij} d\phi^i d\phi^j$. They concluded that in this case the best C_1 is

$$\max_{\mathbf{a}\in\mathbf{R}^d} \min_{\mathbf{m}\in\mathbf{Z}^d} \sqrt{\sum_{i,j} g^{ij}(m_i - a_i)(m_j - a_j)}, \tag{1.2}$$

where $\left[g^{ij}\right]$ is the inverse of the constant positive definite matrix $[g_{ij}]$. For instance, if the metric tensor is diagonal with $g^{ij} = c_i\delta^{ij}$, then (1.2) equals $\sqrt{c_1 + c_2 + \cdots + c_d}/2$.

It is certainly desirable to have an explicit formula for (1.2), in terms of the matrix $\left[g^{ij}\right]$ or its invariants. This problem becomes geometrically intuitive if one views a d-dimensional flat torus as a quotient $\mathbf{R}^d/L$, where L is a lattice in $\mathbf{R}^d$ of maximal rank [**MH**]. If L has basis $\{v_1, v_2, \ldots, v_d\}$ then the metric is given by $g_{ij} = \langle v_i, v_j\rangle$, where $\langle , \rangle$ denotes the standard inner product in $\mathbf{R}^d$. It turns out that for some lattices the Vafa-Witten bound is easy to calculate while for others it is not.

To see just how this distinction arises we will look now at flat metrics on a torus from the viewpoint of homogeneous spaces. The space $\mathrm{Met}(\mathbf{T}^d)$ of flat metrics on $\mathbf{T}^d$ can be identified with the homogeneous space $\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d)$ [**B**] under the transformation

$$\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d) \ni \widehat{\Phi} \longmapsto [g_{ij}] \in \mathrm{Met}(\mathbf{T}^d), \tag{1.3}$$

where if $\Phi \in \mathrm{GL}(d, \mathbf{R})$ then $[g_{ij}]$ is given by

$$g_{ij} := \langle \Phi^{-1}\mathbf{e}_i, \Phi^{-1}\mathbf{e}_j\rangle,$$

$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d)$ being the standard basis in the Euclidean space $\mathbf{R}^d$.

In other words, $[g_{ij}] = \left(\Phi^{-1}\right)^t \Phi^{-1}$, or equivalently $\left[g^{ij}\right] = \Phi\Phi^t$. It follows that under the identification (1.3) the first Vafa-Witten bound becomes

$$\max_{\mathbf{a}\in\mathbf{R}^d} \min_{\mathbf{m}\in\mathbf{Z}^d} \sqrt{\langle \Phi^t(\mathbf{m} - \mathbf{a}), \Phi^t(\mathbf{m} - \mathbf{a})\rangle}. \tag{1.4}$$

It is obvious (see also Proposition 2.2, c)) that a conformal change of the metric $[g_{ij}]$ by a factor r changes (1.2) by a factor of $1/\sqrt{r}$. As a result, it suffices to calculate (1.2) for metrics of fixed volume, or equivalently to replace $\mathrm{GL}(d, \mathbf{R})/\,\mathrm{O}(d)$ with $\mathrm{SL}(d, \mathbf{R})/\,\mathrm{SO}(d)$ in (1.4).

Notice now that (1.4) factors to the double coset space $\mathrm{SL}(d,\mathbf{Z})\backslash\mathrm{SL}(d,\mathbf{R})/\mathrm{SO}(d)$. Indeed, if $\Phi\in\mathrm{SL}(d,\mathbf{R})$ and $\Psi\in\mathrm{SL}(d,\mathbf{Z})$ then, for $\mathbf{a}\in\mathbf{R}^d$,

$$\min_{\mathbf{m}\in\mathbf{Z}^d}\sqrt{\langle(\Psi\Phi)^t(\mathbf{m}-\mathbf{a}),(\Psi\Phi)^t(\mathbf{m}-\mathbf{a})\rangle}=\min_{\mathbf{m}\in\mathbf{Z}^d}\sqrt{\langle\Phi^t\left(\mathbf{m}-\Psi^t\mathbf{a}\right),\Phi^t\left(\mathbf{m}-\Psi^t\mathbf{a}\right)\rangle}.$$

In conclusion, one might be satisfied with calculating (1.2) only for metrics corresponding to a fundamental domain representing the space $\mathrm{SL}(d,\mathbf{Z})\backslash\mathrm{SL}(d,\mathbf{R})/\mathrm{SO}(d)$, such as the Siegel domain [**R**].

This is the first in a series of two papers addressing the problem of finding an explicit formula for the Vafa-Witten bound (1.2). In it we restrict ourselves to two-dimensional tori and work directly with a flat metric $[g_{ij}]$, whose inverse is $g^{11}=A$, $g^{12}=g^{21}=B$, $g^{22}=C$, where A,B,C are real numbers such that $A>0$, $C>0$, and $AC-B^2>0$. The computation of the Vafa-Witten bound in two dimensions is so classical in scope that it can be handled independently within several areas of mathematics: bilinear form theory, lattice theory, modular group theory. We choose to treat the problem using the framework of bilinear forms simply because this is how Vafa and Witten state their result. The lattice and modular group approaches to flat tori do appear, but only indirectly, either in some of the proofs or in the subsequent interpretations and comparisons. The second paper in the series, to appear elsewhere, will be dedicated to higher dimensional tori and will deal only with metrics corresponding to a Siegel domain.

We summarize now our main results, proven below in Theorem 2.5, Theorem 3.8, and Theorem 4.7.

a) If $\min\{A,C\}\geq 2|B|$, then the first Vafa-Witten bound equals

$$\frac{1}{2}\sqrt{\frac{AC(A+C-2|B|)}{AC-B^2}}$$

b) If $\min\{A,C\}<2|B|$, then the transformation (3.3) given in Section 3 below applied to the inverse of the metric tensor a certain number of times, number controlled by the size of $(AC-B^2)/(\min\{A,C\})^2$, reduces the problem to Case *a*).

c) Metrics corresponding to points in the standard fundamental domain F associated to the action of $\mathrm{SL}(2,\mathbf{Z})$ on the Poincaré upper half plane H do satisfy the inequality $\min\{A,C\}\geq 2|B|$, and so Case *a*) applies to them. Arbitrary metrics can then be investigated by noticing that the transformation (3.3) is the basic step of an algorithm that implements the quotient map

$$\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)\longrightarrow\mathrm{SL}(2,\mathbf{Z})\backslash\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2),$$

viewed as a map from H to F.

In addition, we show that the above results still hold if $\min\{A,C\}$ is compared to $|B|$ rather than $2|B|$ (Corollary 3.18).

2. The Particular Case $\min\{A,C\} \geq 2|B|$

Equip the two-dimensional torus $\mathbf{T}^2$ with a flat metric $[g_{ij}]$, whose inverse is $g^{11} = A$, $g^{12} = g^{21} = B$, $g^{22} = C$, where A, B, C are real numbers such that $A > 0$, $C > 0$, and $AC - B^2 > 0$. Then the first Vafa-Witten bound $\lambda_1 = \lambda_1(A, B, C)$ is given by
(2.1)

$$\lambda_1 = \max_{(a_1,a_2)\in\mathbf{R}^2} \min_{(m_1,m_2)\in\mathbf{Z}^2} \sqrt{A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2}$$

In this section we will calculate λ_1 explicitly in the particular case $\min\{A, C\} \geq 2|B|$. We start with some obvious properties of $\lambda_1(A, B, C)$.

Proposition 2.2. — *If $\lambda_1(A, B, C)$ is defined by* (2.1) *then*

a) $\lambda_1(A, B, C)$ *is symmetric in A and C, i.e.,* $\lambda_1(A, B, C) = \lambda_1(C, B, A)$.

b) $\lambda_1(A, B, C) = \lambda_1(A, |B|, C)$

c) If $r > 0$, then $\lambda_1(rA, rB, rC) = \sqrt{r}\lambda_1(A, B, C)$

d) The set of pairs $(a_1, a_2) \in \mathbf{R}^2$ where $\lambda_1(A, B, C)$ occurs intersects $[0, 1]^2$ and is symmetric with respect to the point $(1/2, 1/2)$.

Proof. — Let $f_{A,B,C} : \mathbf{R}^2 \to [0, \infty)$, be given by
(2.3)

$$f_{A,B,C}(a_1, a_2) := \min_{(m_1,m_2)\in\mathbf{Z}^2} \left(A(m_1 - a_1)^2 + 2B(m_1 - a_1)(m_2 - a_2) + C(m_2 - a_2)^2\right)$$

Then the proposition follows from the following properties of $f_{A,B,C}$, respectively.

a) $f_{A,B,C}(a_1, a_2) = f_{C,B,A}(a_2, a_1)$

b) $f_{A,-B,C}(a_1, a_2) = f_{A,B,C}(a_1, -a_2)$

c) If $r > 0$, then $f_{rA,rB,rC} = rf_{A,B,C}$

d) $f_{A,B,C}(a_1 + 1, a_2 + 1) = f_{A,B,C}(a_1, a_2) = f_{A,B,C}(1 - a_1, 1 - a_2)$. □

Remark 2.4. — According to the above proposition in order to find $\lambda_1(A, B, C)$ it is enough to assume that $A \geq C$ and $B \geq 0$ (from *a*) and *b*)), to normalize the metric tensor such that $AC - B^2 = 1$ (from *c*)), and to look for $(a_1, a_2) \in [0, 1]^2$ maximizing $f_{A,B,C}$ only in a suitable "half" of $[0, 1]^2$, for instance $[0, 1] \times [0, 1/2]$ (from *d*).

Theorem 2.5. — *Assume that the torus $\mathbf{T}^2$ is equipped with a flat metric $[g_{ij}] \leftrightarrow (A, B, C)$ such that $\min\{A, C\} \geq 2|B|$. Then the first Vafa-Witten bound is given by the formula*

$$\lambda_1(A, B, C) = \frac{1}{2}\sqrt{\frac{AC(A + C - 2|B|)}{AC - B^2}} \tag{2.6}$$

Proof. — By Proposition 2.2 and Remark 2.4 it suffices to prove Formula 2.6 for $A \geq C \geq 2B \geq 0$ and $AC - B^2 = 1$. As a result, $B^2 \leq 1/3$. The theorem is then equivalent to showing that

$$\max_{(a_1,a_2)\in[0,1]\times[0,1/2]} f_{A,B,C}(a_1, a_2) = \frac{AC(A + C - 2B)}{4}, \tag{2.7}$$

where $f_{A,B,C}$ is the function given by Equation 2.3.

To this end fix $(a_1, a_2) \in [0,1] \times [0,1/2]$. For $(m_1, m_2) \in \mathbf{Z}^2$,

$$\begin{aligned} A(m_1-a_1)^2 &+ 2B(m_1-a_1)(m_2-a_2) + C(m_2-a_2)^2 \\ &= C\left(\frac{AC-B^2}{C^2}(m_1-a_1)^2 + \left(\frac{B}{C}(m_1-a_1)+(m_2-a_2)\right)^2\right) \\ &= \frac{1}{C}(m_1-a_1)^2 + C\left(\frac{B}{C}(m_1-a_1)+(m_2-a_2)\right)^2 \\ &= \frac{1}{C}(m_1-b_1)^2 + C\left(\frac{B}{C}m_1+m_2-b_2\right)^2, \end{aligned}$$

where

$$b_1 = a_1 \quad \text{and} \quad b_2 = \frac{B}{C}a_1 + a_2.$$

Thus,

$$f_{A,B,C}(a_1,a_2) = \min_{(m_1,m_2)\in\mathbf{Z}^2}\left(\frac{1}{C}(m_1-b_1)^2 + C\left(\frac{B}{C}m_1+m_2-b_2\right)^2\right). \tag{2.8}$$

By choosing an integer m_1 such that $|m_1 - b_1| \leq 1/2$, followed by an integer m_2 such that $|\frac{B}{C}m_1 + m_2 - b_2| \leq 1/2$, one sees that

$$f_{A,B,C}(a_1,a_2) \leq \frac{1}{4C} + \frac{C}{4}. \tag{2.9}$$

We claim now that $f_{A,B,C}(a_1,a_2)$ occurs for $(m_1,m_2) \in \{(0,0),(0,1),(1,0)\}$. Indeed, let (m_1^0, m_2^0) be an integer pair where $f_{A,B,C}(a_1,a_2)$ occurs. Then $|m_1^0 - b_1| < 1$, since otherwise (2.8) implies that

$$f_{A,B,C}(a_1,a_2) \geq \frac{1}{C},$$

which in conjunction with (2.9) gives $C^2 \geq 3$. But then $1 = AC - B^2 \geq 3 - 1/3$, a contradiction. Since $b_1 = a_1 \in [0,1]$, it follows that $m_1^0 \in \{0,1\}$.

If $m_1^0 = 0$, then

$$f_{A,B,C}(a_1,a_2) = \frac{b_1^2}{C} + \min_{m_2\in\mathbf{Z}} C(m_2-b_2)^2,$$

and so m_2^0 can be chosen from $\{0,1\}$, since $b_2 = \frac{B}{C}a_1 + a_2 \in [0,1]$.

If $m_1^0 = 1$, then

$$f_{A,B,C}(a_1,a_2) = \frac{(1-b_1)^2}{C} + \min_{m_2\in\mathbf{Z}} C\left(m_2 + \frac{B}{C} - b_2\right)^2,$$

and since $\frac{B}{C} - b_2 = \frac{B}{C}(1-a_1) - a_2 \in [-1/2, 1/2]$, m_2^0 can be taken to be 0. The claim follows.