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FOURTH ORDER EQUATIONS IN CONFORMAL GEOMETRY

by

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Abstract. — In this article we review some recent work on fourth order equations in conformal geometry of three and four dimensions. We discuss an existence result for a Yamabe-type equation in dimension three. We examine a generalization of the Cohn-Vossen inequality to dimension four. Finally, we review an application of the fourth order equation to a fully nonlinear equation in dimension four that involves the Ricci tensor.

***Résumé* (Équations d'ordre quatre en géométrie conforme). —** Dans cet article, nous présentons un travail récent sur des équations d'ordre quatre en géométrie conforme de dimensions trois et quatre. On présente un résultat d'existence d'une équation de type Yamabe en dimension trois. On examine une généralisation de l'inégalité de Cohn-Vossen en dimension quatre. Finalement, nous donnons une application, en dimension quatre, de l'équation d'ordre quatre à une équation non linéaire faisant intervenir le tenseur de Ricci.

1. Introduction

In this article we discuss some new developments in the fourth order equations in conformal geometry of three and four dimensions. We refer the reader to [**CY2**] for a survey of some earlier work in this area.

On a Riemannian manifold (M^n, g) of dimension n, the Laplace Beltrami operator is the natural geometric operator. Under conformal change of metric $g_w = e^{2w}g$, when the dimension is two, Δ_{g_w} is related to Δ_g by the simple formula:

$$\Delta_{g_w}(\varphi) = e^{-2\omega}\Delta_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^2). \tag{1}$$

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In dimension greater than two, similar transformation property continues to hold for a modification of the Laplacian operator called the conformal Laplacian operator $L \equiv -\frac{4(n-1)}{n-2}\Delta + R$ where R is the scalar curvature of the metric. We have

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}\omega} L_g \left(e^{\frac{n-2}{2}\omega} \varphi \right) \tag{2}$$

for all $\varphi \in C^\infty(M)$.

In general, we call a metrically defined operator A conformally covariant of bidegree (a, b), if under the conformal change of metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n). \tag{3}$$

A particularly interesting such operator is a fourth order operator on 4-manifolds discovered by Paneitz [**Pa**] in 1983:

$$P\varphi \equiv \Delta^2 \varphi + \delta \left(\frac{2}{3} R I - 2\,\mathrm{Ric} \right) d\varphi \tag{4}$$

where δ denotes the divergence, d the de Rham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P is conformal covariant of bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4w} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4). \tag{5}$$

A fourth order curvature invariant $Q = \frac{1}{12}\{-\Delta R + R^2 - 3|Rc|^2\}$ is associated to the Paneitz operator:

$$Pw + 2Q = 2Q_w e^{4w}.$$

In dimension four, the Paneitz equation has close connection with the Chern-Gauss-Bonnet formula. For a compact oriented 4-manifold,

$$\chi(M) = \frac{1}{4\pi^2} \int_M (\frac{|W|^2}{8} + Q) dV \tag{6}$$

where $\chi(M)$ denotes the Euler characteristic of the manifold M, and $|W|^2$= norm squared of the Weyl tensor. Since $|W|^2 dV$ is a pointwise invariant under conformal change of metric, QdV is the term which measures the conformal change in formula (6).

For a 4-manifold with boundary, [**CQ**] defines a third order boundary operator P_3 which is conformally covariant of bidegree $(1, 3)$:

$$P_3 = -\frac{1}{2}\frac{\partial}{\partial n}\Delta - \tilde{\Delta}\frac{\partial}{\partial n} - \frac{2}{3} H \tilde{\Delta} + L_{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta + \big(\frac{1}{3}R - R_{\alpha N \alpha N}\big)\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla} H \cdot \tilde{\nabla} \tag{7}$$

where ∂n is the unit interior normal, $\tilde{\Delta}$ is the boundary Laplacian, H is the mean curvature, $L_{\alpha\beta}$ the second fundamental form, and $\tilde{\nabla}$ the boundary gradient. The boundary P_3 operator defines the third order curvature invariant T through the equation:

$$-P_3 w + T_w e^{3w} = T \quad \text{on} \quad \partial M \tag{8}$$

where

$$T=\frac{1}{12}\frac{\partial}{\partial n}R+\frac{1}{6}RH-R_{\alpha N\beta N}L_{\alpha\beta}+\frac{1}{9}H^3-\frac{1}{3}\operatorname{Tr}L^3-\frac{1}{3}\tilde{\Delta}H. \tag{9}$$

For 4-manifolds with boundary, the Chern-Gauss-Bonnet formula is supplemented by

$$\chi(M)=\frac{1}{4\pi^2}\int_M\Big(\frac{|W|^2}{8}+Q\Big)dV+\frac{1}{4\pi^2}\int_{\partial M}(L+T)d\Sigma \tag{10}$$

where $Ld\sigma$ is a pointwise conformal invariant of the boundary.

In order to find geometric interpretation for the fourth order invariant Q, we formulated an analogue ([**CQY1**]) of the Cohn-Vossen inequality for complete surfaces with finite total curvature and derived ([**CQY2**]) a compactification criteria for conformally flat 4-manifold using the curvature invariant Q and the assumption of geometric finiteness.

In general dimensions different from four there is also a natural fourth order operator P, which enjoys the conformal covariance property with respect to conformal changes in metrics. The relation of this operator to the Paneitz operator in dimension four is completely analogous to the relation of the conformal Laplacian to the Laplacian in dimension two. On (M^n,g) when $n\neq 4$, define

$$P=(-\Delta)^2+\delta(a_nR+b_n\operatorname{Ric})d+\frac{n-4}{2}Q$$

where

$$Q=c_n|\operatorname{Ric}|^2+d_nR^2-\frac{1}{2(n-1)}\Delta R$$

and

$$a_n=\frac{(n-2)^2+4}{2(n-1)(n-2)},\ b_n=-\frac{4}{n-2},\ c_n=-\frac{2}{(n-2)^2},\ d_n=\frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$$

are dimensional constants. Then (Branson [**Br**]), writing $g_u=u^{\frac{4}{n-4}}g$, $n\neq 4$ we have

$$(P)_u(\varphi)=u^{-\frac{n+4}{n-4}}P(u\varphi) \tag{11}$$

for all $\varphi\in C^\infty(M^n)$. We also have the analogue for the Yamabe equation:

$$Pu=\frac{n-4}{2}Qu^{\frac{n+4}{n-4}}\qquad \text{on}\quad M^n,\quad n\neq 4. \tag{12}$$

Such semilinear biharmonic equations with critical exponents were first investigated by Pucci-Serrin in [**PuS**], they obtained the analogue of the Brezis-Nirenberg result ([**BN**]) in dimensions $n=5,6,7$ for domains in R^n. In the article [**DHL**] there are some criteria for existence for equations of Paneitz type.

It is interesting to note that in dimension three, the equation takes a special form

$$Pu=-\frac{1}{2}Qu^{-7} \tag{13}$$

for the conformal factor $g=u^{-4}g_0$. It is natural to ask whether one can solve the analogue of the Yamabe equation for this operator. In [**XY**] we were able to formulate a criteria for positivity of the operator P in dimension three and obtained some existence result for the equation of prescribing constant Q. The study of this equation is still in a primitive stage, there is much that remains to be developed.

In dimension four, the theory of the fourth order equation can be applied to the study of fully nonlinear equations involving the symmetric functions of the modified Ricci tensor. This set of equations is studied by Viaclovsky [**V**] in his thesis. In dimension four, we can use the fourth order equation as a regularization of the second order equation of prescribing the second elementary symmetric functions $\sigma_2(A)$ where A is the conformal Ricci tensor $A = Rc - \frac{1}{6}Rg$. As a consequence, we were able to give a simple criteria for existence, in a given four dimensional conformal class, of a metric with strongly positive Ricci tensor. The conformal classes in four dimension that satisfy the conformally invariant conditions $\int \sigma_2(A)dV > 0$ and having positive Yamabe invariant, include the 4-sphere, connected sums of up to three copies of $\mathbb{CP}^2$, connected sums of $\mathbb{CP}^2$ with up to eight copies of $\mathbb{CP}^2$ with reversed orientation, and connected sums of up to two copies of $S^2 \times S^2$.

We give an outline of the rest of the paper. In section two we study the fourth order equation on 3-manifolds. We discuss the uniqueness question for the equation (12) in Euclidean 3-space. We formulate a criteria for existence result for prescribing constant Q for a class of 3-manifolds. In section three, we consider the fourth order equation on conformally flat 4-manifolds and report on the compactification criteria of [**CQY2**]. Finally in section four we discuss the fully nonlinear equations for prescribing the elementary symmetric functions of the conformal Ricci tensor on a 4-manifold.

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2. The fourth order operator in dimension three

For the P operator in dimension three we have

$$P = (-\Delta)^2 + \delta(\frac{5}{4}Rg - 4Rc)d - \frac{1}{2}Q \tag{14}$$

where

$$Q = -2|Rc|^2 + \frac{23}{32}R^2 - \frac{1}{4}\Delta R. \tag{15}$$

The Q curvature equation is given by

$$Pu = -\frac{1}{2}Qu^{-7}. \tag{16}$$

The analogue of the Yamabe problem in this setting would be to solve equation (16) with Q given by a constant. This is naturally the Euler equation of the variational functional

$$F[u] = \Big(\int_M u^{-6} dV\Big)^{1/3} \int_M Pu \cdot u dV. \tag{17}$$

The direct method would be to minimize the functional over the class of positive functions in the Sobolev space $W^{2,2}$. The negative exponent in the integral means that the analytic difficulty is associated with the conformal factor touching zero. The negative sign of the coefficient for the Q curvature term in equation (16) makes a sharp contrast with the Yamabe equation. For example, among the eight standard geometries, only in the case of the sphere and hyperbolic 3-manifolds the Q curvature is positive. There is some preliminary result in this direction.

In studying a nonlinear equation involving a critical exponent, it will be important to have an understanding of the blowup solutions. Thus one is interested in global positive solutions in Euclidean 3-space of the equation

$$\Delta^2 u = -\frac{15}{16} u^{-7}. \tag{18}$$

Assuming the solution actually came from a positive solution of the corresponding equation on S^3 via the stereographic projection, it would have the natural asymptotic behavior: $u(x)/|x|$ tends to a positive constant as $|x|$ tends to infinity. Adapting the method of moving planes, Choi and Xu ([**CX**]) has classified such entire solutions: after translations and dilations u is of the form $u(x) = 2^{-1/2}(1+|x|^2)^{1/2}$. In the same article, they also showed that the same assertion holds if, instead of the asymptotic condition at infinity, the scalar curvature of the metric is assumed to be non-negative at infinity.

The question of existence turns out to be simplest when the operator P is positive and the manifold (M^3, g_0) is in the positive Yamabe class. We have

***Theorem 2.1* ([XY]).** — *If (M^3, g_0) has positive scalar curvature and the operator P is positive, then the functional F achieves a positive minimum at a positive smooth function u.*

Remark 2.1

1. The positivity of the operator P does not follow from the positivity of the scalar curvature. In fact on the standard 3-sphere the operator P has a negative eigenvalue due to the fact $Q_0 = 15/8$. A simple criteria for positivity of the operator P on (M^3, g) is that there is a conformal metric in which $Q < 0$ and $R > 0$. The class of conformal structures satisfying the these conditions includes the standard product structures on $S^1 \times S^2$ and their connected sums. In view of Yau's conjecture [**SY**], it is quite likely that the only possible topology supporting conformal structures with these positivity conditions are those listed.

2. In a recent article, Djadli-Hebey-Ledoux [**DHL**] studied the best constants in a Sobolev inequality related to the Paneitz equation in dimensions $n \geq 5$.