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# SOME LOCAL AND NON-LOCAL VARIATIONAL PROBLEMS IN RIEMANNIAN GEOMETRY

*by*

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***Abstract.*** **—** In this article we will give a brief summary of some recent work on two variational problems in Riemannian geometry. Although both involve the study of elementary symmetric functions of the eigenvalues of the Ricci tensor, as far as technique and motivation are concerned the problems are actually quite different.

***Résumé*** **(Problèmes variationnels locaux et non-locaux en géométrie riemannienne)**
Dans cet article nous donnons un aperçu d'un travail récent sur deux problèmes variationnels en géométrie riemannienne. Bien que les deux problèmes soient basés sur l'étude des fonctions symétriques élémentaires des valeurs propres du tenseur de Ricci, les techniques et les motivations sont en réalité différentes.

*For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.* –Leonhard Euler

## 1. Quadratic Riemannian functionals

The first problem we will discuss represents joint work of the author with Jeff Viaclovsky ([**GV00**]). To describe it, let us begin with some general notions.

Let $M$ be a smooth manifold, $\mathcal{M}$ the space of smooth Riemannian metrics on $M$, and $\mathcal{G}$ the diffeomorphism group of $M$. A functional $F : \mathcal{M} \to \mathbb{R}$ is called *Riemannian* if $F$ is invariant under the action of $\mathcal{G}$; i.e., if $F(\phi^* g) = F(g)$ for each $\phi \in \mathcal{G}$ and $g \in \mathcal{M}$. If we endow $\mathcal{M}$ with a natural $L^2$–Sobolev norm, then we may speak of *differentiable* Riemannian functionals. Letting $S_2(M)$ denote the bundle of symmetric two–tensors, we then say that $F : \mathcal{M} \to \mathbb{R}$ has a *gradient at* $g \in \mathcal{M}$ if $\frac{d}{dt} F[g + th]|_{t=0} = \int g(h, \nabla F)\, d\,\mathrm{vol}_g$ for some $\nabla F \in \Gamma(S_2(M))$ and all $h \in \Gamma(S_2(M))$.

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An important example of a Riemannian functional is of course the total scalar curvature

$$S[g] = \int R_g \, d\,\mathrm{vol}_g \tag{1.1}$$

where $R_g$ denotes the scalar curvature of $g$. For Riemannian geometers, the importance of (1.1) lies in the fact that when $M$ is compact, critical points of $S|_{\mathcal{M}_1}$, where

$$\mathcal{M}_1 = \{g \in \mathcal{M} | \ \mathrm{Vol}(g) = 1\},$$

are Einstein (see [**Bes87**]). In the Lorentzian setting, Hilbert showed that the equations of general relativity can be realized in a similar manner ([**Hil72**]).

Our interest here is in functionals that are obtained by integrating a polynomial which is quadratic in the curvature. By Weyl's invariant theory ([**Wey39**]), a basis for these functionals is

$$\begin{aligned} \mathcal{R}[g] = \int |\,\mathrm{Riem}_g\,|^2 \, d\,\mathrm{vol}_g, \quad \rho[g] = \int |\,\mathrm{Ric}_g\,|^2 \, d\,\mathrm{vol}_g, \\ \tau[g] = \int R_g^2 \, d\,\mathrm{vol}_g, \end{aligned} \tag{1.2}$$

where $\mathrm{Riem}_g$ and $\mathrm{Ric}_g$ denote respectively the Riemann curvature tensor and Ricci curvature tensor of $g$. Such functionals arise in certain field theories in physics; in particular $\mathcal{R}$ can be viewed as a Riemannian analogue of Yang–Mills (see [**Bac21**], [**Bou96**], [**Bes87**]).

From the variational point of view, the functionals in (1.2) have the apparent advantage of being bounded below, and thus more amenable to the direct method. However, the associated Euler equations are quite complicated (see [**And97**], [**Bes87**], [**Lam98**]). Indeed, in [**Lam98**] a critical point of $\mathcal{R}$ is constructed on $S^3$ which does not have constant sectional curvature. Thus, even if successful, it is not clear that such an approach would yield Einstein metrics (under certain geometric and topological constraints there are some exceptions; see [**Gur98**]).

Before we give an exact description of the functional we will be interested in, for the purpose of motivation it may be helpful to first recall a basic fact about the decomposition of the curvature tensor (see [**Bes87**]). Let $\odot$ denote the Kulkarni-Nomizu product, and define the tensor $C_g = \mathrm{Ric} - \frac{R}{2(n-1)} g$. Then the full curvature tensor of $g$ can be decomposed as

$$\mathrm{Riem} = W + \frac{1}{(n-2)} C \odot g,$$

where $W$ denotes the Weyl curvature tensor of $g$. In three dimensions, we have $C_g = \mathrm{Ric} - \frac{R}{4} g$, and the Weyl tensor vanishes. Thus, the full curvature tensor is actually determined by $C_g$.

Now if $\sigma_k : \mathbb{R}^3 \to \mathbb{R}$ denote the elementary symmetric functions, then the scalar curvature can be expressed as $R = 4\sigma_1(C)$. It follows that the natural quadratic counterpart to (1.1) is the functional:

$$\mathcal{F}_2[g] = \int_M \sigma_2\left(C_g\right)\, d\,\mathrm{vol}_g\,. \tag{1.3}$$

A simple calculation gives

$$\mathcal{F}_2[g] = \int_M \left(-\frac{1}{2}|\,\mathrm{Ric}\,|^2 + \frac{3}{16}R^2\right) d\,\mathrm{vol}_g\,. \tag{1.4}$$

$\mathcal{F}_2$ is therefore quadratic in the curvature of $g$, and is a non–convex linear combination of the functionals in (1.2).

There are interesting parallels between the functionals $S$ and $\mathcal{F}_2$. Like the total scalar curvature, $\mathcal{F}_2$ is neither bounded above nor below on $\mathcal{M}_1$. Further, one can consider a constrained version of $\mathcal{F}_2$ by restricting to a fixed conformal class; see [**Viaa**], [**Viab**]. In these works, the Euler equation for $\mathcal{F}_2|_{[g]_1}$, where

$$[g]_1 = \{\tilde{g} = e^{2w}g, w \in C^\infty(M, \mathbf{R})|\ \mathrm{Vol}(\tilde{g}) = 1\},$$

is shown to be $\sigma_2(\mathrm{Ric} - \frac{1}{4}Rg) \equiv \lambda =$ constant. Remarkably, this scalar equation encodes information about the sectional curvatures of $g$, provided $\lambda > 0$:

***Proposition 1.1***. — *Let $M$ be three-dimensional. If $\sigma_2(\mathrm{Ric} - \frac{1}{4}Rg)_x > 0$ then the sectional curvatures of $g$ at $x$ are either all positive or all negative. In particular, critical points of $\mathcal{F}_2|_{[g]_1}$ with $\mathcal{F}_2[g] > 0$ have either strictly positive or strictly negative sectional curvature.*

Moreover, we have the following new characterization of (compact) Einstein three-manifolds:

***Theorem 1.1* ([GV00])**. — *Let $M$ be compact and three–dimensional. Then a metric $g$ with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if $g$ has constant sectional curvature.*

***Remark***

1. The condition $\mathcal{F}_2[g] \geq 0$ in Theorem 1.1 is necessary: if $E = \mathrm{Ric} - \frac{1}{3}Rg$ denotes the trace–free Ricci tensor, then

$$\sigma_2(C) = \sigma_2\left(\mathrm{Ric} - \frac{1}{4}Rg\right) = -\frac{1}{2}|E|^2 + \frac{1}{48}R^2. \tag{1.5}$$

   Thus, if $g$ has constant curvature, $\sigma_2 = \frac{1}{48}R^2 \geq 0$.
2. The condition $\mathcal{F}_2[g] > 0$ may be thought of as an *ellipticity* assumption. To our knowledge, this is the first example of a quadratic Riemannian functional in three dimensions whose elliptic critical points are necessarily of constant curvature.
3. The case $\mathcal{F}_2[g] = 0$ is the case of degenerate ellipticity, and the proof in this case is much more delicate, as the curvature may change sign.

4. When $\mathcal{F}_2[g] < 0$, we have left the region of ellipticity, and we do not expect a simple classification of these critical points. Indeed, the construction of [**Lam98**] provides an example of a critical metric on $S^3$ with $\mathcal{F}_2[g] < 0$.

In [**GV00**] we also considered a constrained version of the problem: $\mathcal{F}_2|_{\Xi}$, where

$$\Xi = \left\{ g \in \mathcal{M}_1 \middle| \ \sigma_2(C_g) = \int_M \sigma_2(C_g)\, d\,\mathrm{vol}_g > 0, \text{ and } R_g < 0 \right\}.$$

In analogy with the work of Koiso for the scalar curvature (see [**Koi79**]), one can show that $\Xi$ is in fact a submanifold of $\mathcal{M}_1$. Restricting to $\Xi$ introduces a Lagrange multiplier term into the Euler equation, and like the corresponding problem for the scalar curvature we can show that this term vanishes:

***Theorem 1.2*** **([GV00])**. — *Let $M$ be compact and three–dimensional. If $g$ is a critical point of $\mathcal{F}_2|_{\Xi}$, then $g$ is hyperbolic.*

The proof of Theorem 1.1 naturally divides into two cases: first, assuming the critical metric $g$ has $\sigma_2(C) > 0$, then the more difficult case of $\sigma_2(C) = 0$. The former case further divides into two parts, according to whether the scalar curvature is strictly positive or strictly negative.

The Euler equation for $\mathcal{F}_2$ is quite complicated; see [**GV00**] for a detailed account of the first variation. The precise formula is:

$$\begin{aligned}(\nabla\mathcal{F}_2)_{ij} &= \frac{1}{2}\Delta E_{ij} + \frac{1}{24}\Delta R g_{ij} - \frac{1}{8}\nabla_i\nabla_j R \\ &\qquad - 2E_{im}E_{mj} - \frac{5}{24}RE_{ij} + \frac{1}{36}R^2 g_{ij} - \frac{3}{2}\sigma_2(C)g_{ij}.\end{aligned} \tag{1.6}$$

For the proof of the case when $\sigma_2(C) > 0$ and $R > 0$ it will be helpful to introduce the tensor $T = -\,\mathrm{Ric} + \frac{1}{2}Rg$. The significance of $T$ is the following: suppose $\Pi$ is a non-degenerate tangent plane in $T_pM$ for some $p \in M$. If $u \in T_pM$ is a unit normal to $\Pi$, then the sectional curvature of $\Pi$ is $T(u,u)$. In particular, if $\sigma_2(C) > 0$ and $R > 0$ then by Proposition 1.1 the tensor $T$ is positive definite. In fact, the same argument shows that when $R > 0$ but $\sigma_2(C) \geq 0$, then $T$ is positive semi-definite.

Now suppose that $g$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$. Taking the inner product with $E$ on both sides of (1.6) we get

$$\begin{aligned}\frac{1}{4}T^{ij}\nabla_i\nabla_j R = \Delta\sigma_2(C) + |\nabla E|^2 - \frac{1}{24}|\nabla R|^2 \\ + 4\,\mathrm{tr}\, E^3 + \frac{5}{12}R|E|^2 + 2g(\nabla\mathcal{F}_2, E),\end{aligned} \tag{1.7}$$

where $\mathrm{tr}\, E^3 = E_i^j E_j^k E_k^i$. Since $g$ is critical, $\nabla\mathcal{F}_2 = 0$ and $\Delta\sigma_2(C) = 0$, so

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R = |\nabla E|^2 - \frac{1}{24}|\nabla R|^2 + 4\,\mathrm{tr}\, E^3 + \frac{5}{12}R|E|^2. \tag{1.8}$$

To show that $E = 0$ when $\sigma_2(C) > 0$ and $R > 0$ we use the maximum principle, which requires the following lemma:

***Lemma 1.1***. — *Suppose $g$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ and $\sigma_2(C) \geq 0$. Let $U \subset M$ be an open set on which $R > 0$. Then in $U$,*

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R \geq \frac{1}{\sqrt{6}}|E|^3. \tag{1.9}$$

*Proof*. — Since $\sigma_2(C)$ is a non-negative constant, it is easy to see that

$$|\nabla E|^2 \geq \frac{1}{24}|\nabla R|^2. \tag{1.10}$$

If we substitute this into (1.8) we obtain

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R \geq 4\,\mathrm{tr}\,E^3 + \frac{5}{12}R|E|^2. \tag{1.11}$$

Using the sharp inequality

$$\mathrm{tr}\,E^3 \geq -\frac{1}{\sqrt{6}}|E|^3, \tag{1.12}$$

we conclude

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R \geq -\frac{4}{\sqrt{6}}|E|^3 + \frac{5}{12}R|E|^2. \tag{1.13}$$

Since $\sigma_2(C) \geq 0$, we have $R \geq 2\sqrt{6}|E|$, thus

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R \geq -\frac{4}{\sqrt{6}}|E|^3 + \frac{5}{12}2\sqrt{6}|E|^3 = \frac{1}{\sqrt{6}}|E|^3. \tag{1.14}$$

□

Now if $\sigma_2(C) > 0$ and $R > 0$ on $M$, then we can apply Lemma 1.1 on $U = M$. Since $T > 0$, we conclude by the maximum principle that $E \equiv 0$ on $M$.

The case where $\sigma_2(C) > 0$ and $R < 0$ requires a different idea. The argument in ([**GV00**]) is very much inspired by the work of Koiso ([**Koi78**]) and Bourguignon ([**Bou81**]). Here we will offer a different (but equivalent) argument which seems more natural, in part because it sheds some light on the rather roccoco expression for the gradient in (1.6).

Note that the tensor $C$, being a section of $S_2(M)$, can alternatively be viewed as a one-form with values in the cotangent bundle $T^*M$. We will write this as $C \in \Omega^1(T^*M)$. Now consider the complex

$$\Omega^0(T^*M) \to \Omega^1(T^*M) \to \Omega^2(T^*M) \to \cdots \tag{1.15}$$

The Riemannian connection $\nabla : \Omega^0(T^*M) \to \Omega^1(T^*M)$, and induces the exterior derivative $d^\nabla : \Omega^1(T^*M) \to \Omega^2(T^*M)$. We also have the adjoint maps $\delta^\nabla : \Omega^2(T^*M) \to \Omega^1(T^*M)$ and $\nabla^* : \Omega^1(T^*M) \to \Omega^0(T^*M)$. Note that $\nabla^*$ is just the usual divergence operator on symmetric two-tensors. Moreover, a manifold is locally conformally flat if and only if the tensor C satisfies $d^\nabla C \equiv 0$.