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DEPENDENCE OF THE DIRAC SPECTRUM ON THE SPIN STRUCTURE

by

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Abstract. — The theme is the influence of the spin structure on the Dirac spectrum of a spin manifold. We survey examples and results related to this question.

Résumé **(Dépendance du spectre de l'opérateur de Dirac par rapport à la structure spinorielle)**
Sur une variété spinorielle, nous étudions la dépendance du spectre de l'opérateur de Dirac par rapport à la structure spinorielle. Nous donnons ensuite un résumé des exemples et des résultats liés à cette question.

1. Introduction

The relation between the geometry of a Riemannian manifold and the spectrum of its Laplace operator acting on functions (or more generally, on differential forms), has attracted a lot of attention. This is the question how shape and sound of a space are related. A beautiful introduction into this topic can be found in [**12**]. When one passes from this "bosonic" theory to "fermions", i.e. when turning to spinors and the Dirac operator, a new object enters the stage, the *spin structure*. This is a global topological object needed to define spinors. The question arises how this piece of structure, in addition to the usual geometry of the manifold, influences the spectrum of the Dirac operator.

It has been known for a long time that even on the simplest examples such as the 1-sphere the Dirac spectrum does depend on the spin structure. We will discuss the 1-sphere, flat tori, 3-dimensional Bieberbach manifolds, and spherical space forms in some detail. For these manifolds the spectrum can be computed explicitly. For some of these examples an important invariant computed out of the spectrum, the η-invariant, also depends on the spin structure. On the other hand, under a certain assumption,

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the difference between the η-invariants for two spin structures on the same manifold must be an integer. Hence the two η-invariants are not totally unrelated.

We also look at circle bundles and the behavior of the Dirac spectrum under collapse. This means that one shrinks the fibers to points. The spin structure determines the qualitative spectral behavior. If the spin structure is projectable, then some eigenvalues tend to $\pm\infty$ while the others essentially converge to the eigenvalues of the basis manifold. If the spin structure if nonprojectable, then all eigenvalues diverge.

In most examples it is totally hopeless to try to explicitly compute the Dirac (or other) spectra. Still, eigenvalue estimates are very often possible. So far, these estimates have not taken into account the spin structure despite its influence on the spectrum. The reason for this lies in the essentially local methods such as the Bochner technique. In order to get better estimates taking the spin structure into account one first has to find new, truly spin geometric invariants. We discuss some of the first steps in this direction. Here the *spinning systole* is the relevant spin geometric input.

Finally we look at noncompact examples in order to check if the continuous spectrum is affected by a change of spin structure. It turns out that this is the case. There are hyperbolic manifolds having two spin structures such that for the first one the Dirac spectrum is discrete while it is all of $\mathbb{R}$ for the other one. The influence of the spin structure could hardly be any more dramatic.

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2. Generalities

Let us start by collecting some terminology and basic facts. A more thorough introduction to the concepts of spin geometry can e.g. be found in [**15, 9, 18**]. Let M denote an n-dimensional oriented Riemannian manifold with a spin structure P. This is a $\mathrm{Spin}(n)$-principal bundle which doubly covers the bundle of oriented tangent frames $P_{\mathrm{SO}}M$ of M such that the canonical diagram

$$\begin{array}{ccccc}
P \times \mathrm{Spin}(n) & \longrightarrow & P & & \\
\downarrow & & \downarrow & \searrow & \\
P_{\mathrm{SO}}M \times \mathrm{SO}(n) & \longrightarrow & P_{\mathrm{SO}}M & \longrightarrow & M
\end{array}$$

commutes. Such a spin structure need not exist, e.g. complex projective plane $\mathbb{CP}^2$ has none. If M has a spin structure we call M a *spin manifold.* The spin structure of a spin manifold is in general not unique. More precisely, the cohomology $H^1(M;\mathbb{Z}_2)$ of a spin manifold acts simply transitively on the set of all spin structures.

Given a spin structure P one can use the spinor representation

$$\mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$$

to construct the associated *spinor bundle* ΣM over M. Here Σ_n is a Hermitian vector space of dimension $2^{[n/2]}$ on which $\mathrm{Spin}(n)$ acts by unitary transformations. Hence ΣM is a Hermitian vector bundle of rank $2^{[n/2]}$. Sections in ΣM are called *spinor fields* or simply *spinors*. Note that unlike differential forms the definition of spinors requires the choice of a spin structure. The Levi-Civita connection on $P_{\mathrm{SO}}M$ can be lifted to P and therefore induces a covariant derivative ∇ on ΣM.

Algebraic properties of the spinor representation ensure existence of *Clifford multiplication*

$$T_pM \otimes \Sigma_pM \to \Sigma_pM, \quad X \otimes \psi \mapsto X \cdot \psi,$$

satisfying the relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi + 2\langle X, Y\rangle \psi = 0$$

for all $X, Y \in T_pM$, $\psi \in \Sigma_pM$, $p \in M$. Here $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric.

The *Dirac operator* acting on spinors is defined as the composition of ∇ with Clifford multiplication. Equivalently, if $e_1, \ldots, e_n$ is an orthonormal basis of T_pM, then

$$(D\psi)(p) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi.$$

The Dirac operator is a formally self-adjoint elliptic differential operator of first order. If the underlying Riemannian manifold M is complete, then D, defined on compactly supported smooth spinors, is essentially self-adjoint in the Hilbert space of square-integrable spinors. General elliptic theory ensures that the spectrum of D is discrete if M is compact and satisfies Weyl's asymptotic law

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^n} = \frac{2^{[n/2]} \cdot \mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2} + 1\right)},$$

where $N(\lambda)$ is the number of eigenvalues whose modulus is $\leq \lambda$. This implies that the series

$$\eta(s) = \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}$$

converges for $s \in \mathbb{C}$ if the real part of s is sufficiently large. Here summation is taken over all nonzero eigenvalues λ of D, each eigenvalue being repeated according to its multiplicity. It can be shown that the function $\eta(s)$ extends to a meromorphic function on the whole complex plane and has no pole at $s = 0$. Evaluation of this meromorphic extension at $s = 0$ gives the η-invariant,

$$\eta := \eta(0).$$

If M is complete but noncompact, then D may also have eigenvalues of infinite multiplicity, cumulation points of eigenvalues, and continuous spectrum.

3. The baby example

In order to demonstrate the dependence of the Dirac spectrum on the choice of spin structure the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ can serve as a simple but nonetheless illustrative example. Since the frame bundle $P_{\mathrm{SO}}S^1$ is trivial we can immediately write down the trivial spin structure $P = S^1 \times \mathrm{Spin}(1)$. Note that $\mathrm{Spin}(1) = \mathbb{Z}_2$ and $\Sigma_1 = \mathbb{C}$. The associated spinor bundle is then also trivial and 1-dimensional. Hence spinors are simply $\mathbb{C}$-valued functions on S^1. The Dirac operator is nothing but

$$D = i\frac{d}{dt}.$$

Elementary Fourier analysis shows that the spectrum consists of the eigenvalues

$$\lambda_k = k$$

with corresponding eigenfunctions $t \mapsto e^{-ikt}$, $k \in \mathbb{Z}$. Since the spectrum is symmetric about zero, the η-series, and in particular, the η-invariant vanishes,

$$\eta = 0.$$

From $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$ we see that S^1 has a second spin structure. It can be described as $\tilde{P} = ([0,2\pi] \times \mathrm{Spin}(1))/\sim$ where $\sim$ identifies 0 with 2π while it interchanges the two elements of Spin(1). Let us call this spin structure the *nontrivial spin structure* of S^1. Spinors with respect to this spin structure no longer correspond to functions on S^1, i.e. to 2π-periodic functions on $\mathbb{R}$, but rather to 2π-anti-periodic complex-valued functions on $\mathbb{R}$,

$$\psi(t + 2\pi) = -\psi(t).$$

This time the eigenvalues are

$$\lambda_k = k + \frac{1}{2},$$

$k \in \mathbb{Z}$, with eigenfunctions $t \mapsto e^{-i(k+\frac{1}{2})t}$. Again, the spectrum is symmetric about 0, hence $\eta = 0$. Vanishing of the η-invariant is in fact not surprising. One can show that always $\eta = 0$ for an n-dimensional manifold unless $n \equiv 3 \bmod 4$.

The example S^1 has shown that the eigenvalues of the Dirac operator definitely do depend on the choice of spin structure. Even the dimension of the kernel of the Dirac operator is affected by a change of spin structure. For the trivial spin structure of S^1 it is 1 while it is zero for the nontrivial spin structure.

We conclude this section with a remark on extendability of spin structures because this sometimes causes confusion. If M is a Riemannian spin manifold with boundary ∂M, then a spin structure on M induces one on ∂M. To see this consider the frame bundle $P_{\mathrm{SO}}\partial M$ of the boundary as a subbundle of $P_{\mathrm{SO}}M$ restricted to the boundary by completing a frame for ∂M with the exterior unit normal vector to a frame for M. Now the inverse image of $P_{\mathrm{SO}}\partial M$ under the covering map $P \to P_{\mathrm{SO}}M$ defines a spin structure on ∂M.

Look at the case that M is the disc with S^1 as its boundary. Since the disk is simply connected it can have only one spin structure. Hence only one of the two spin structures of S^1 extends to the disc. The tangent vector to the boundary S^1 together with the unit normal vector forms a frame for the disk which makes one full rotation when going around the boundary one time. It is therefore a loop in the frame bundle of the disk whose lift to the spin structure does not close up. Thus the induced spin structure on the boundary is the nontrivial spin structure of S^1 while the trivial spin structure does not bound. Hence from a cobordism theoretical point of view the trivial spin structure is nontrivial and vice versa.

4. Flat tori and Bieberbach manifolds

The case of higher-dimensional flat tori is very similar to the 1-dimensional case. There are 2^n different spin structures on $T^n = \mathbb{R}^n/\Gamma$ where Γ is a lattice in $\mathbb{R}^n$. Let $b_1, \dots, b_n$ be a basis of Γ, let $b_1^*, \dots, b_n^*$ be the dual basis for the dual lattice Γ^*. Spin structures can then be classified by n-tuples $(\delta_1, \dots, \delta_n)$ where each $\delta_j \in \{0, 1\}$ indicates whether or not the spin structure is twisted in direction b_j. The spectrum of the Dirac operator can then be computed:

Theorem 4.1 **(Friedrich [14])**. — *The eigenvalues of the Dirac operator on $T^n = \mathbb{R}^n/\Gamma$ with spin structure corresponding to $(\delta_1, \dots, \delta_n)$ are given by*

$$\pm 2\pi \left| b^* + \frac{1}{2} \sum_{j=1}^{n} \delta_j b_j^* \right|$$

where b^* runs through Γ^* and each b^* contributes multiplicity $2^{[n/2]-1}$.

Again the spectrum depends on the choice of spin structure. In particular, eigenvalue 0 occurs only for the trivial spin structure given by $(\delta_1, \dots, \delta_n) = (0, \dots, 0)$. Since again the spectrum is symmetric about zero, the η-invariant vanishes, $\eta = 0$, for all spin structures.

This changes if one passes from tori to more general compact connected flat manifolds, also called *Bieberbach manifolds.* They can always be written as a quotient $M = G \backslash T^n$ of a torus by a finite group G. In three dimensions, $n = 3$, there are 5 classes of compact oriented Bieberbach manifolds besides the torus. Their Dirac spectra have been calculated by Pfäffle [**20**] for all flat metrics. This time one finds examples with asymmetric spectrum and the η-invariant depends on the choice of spin structure.

Theorem 4.2 **(Pfäffle [20])**. — *The η-invariant of the 3-dimensional compact oriented Bieberbach manifolds besides the torus are given by the following table:*