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# CURVATURE AND SMOOTH TOPOLOGY IN DIMENSION FOUR

*by*

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***Abstract.*** — Seiberg-Witten theory leads to a delicate interplay between Riemannian geometry and smooth topology in dimension four. In particular, the scalar curvature of any metric must satisfy certain non-trivial estimates if the manifold in question has a non-trivial Seiberg-Witten invariant. However, it has recently been discovered [**26**, **27**] that similar statements also apply to other parts of the curvature tensor. This article presents the most salient aspects of these curvature estimates in a self-contained manner, and shows how they can be applied to the theory of Einstein manifolds. We then probe the issue of whether the known estimates are optimal by relating this question to a certain conjecture in Kähler geometry.

***Résumé* (Courbure et topologie lisse en dimension 4).** — La théorie de Seiberg-Witten révèle des liens étonnants entre la géométrie riemannienne et la topologie lisse en dimension 4. En particulier, sur une variété compacte dont un invariant Seiberg-Witten ne s'annule pas, la norme de la courbure scalaire est minorée, d'une manière uniforme et non triviale, pour toute métrique riemannienne. Cependant, on a récemment démontré [**26**, **27**] des estimées analogues à l'égard de la courbure de Weyl. Dans cet article, nous rendrons compte de ces estimées de courbure, y compris leurs conséquences pour la théorie des variétés d'Einstein. Nous finissons par un examen du problème d'optimalité des estimées actuelles, en reliant cette question à une conjecture en géométrie kählérienne.

## 1. Introduction

In 1994, Witten [**39**] shocked the mathematical world by announcing that the differential-topological invariants of Donaldson [**9**] are intimately tied to the scalar curvature of Riemannian 4-manifolds. His central discovery was a new family of 4-manifold invariants, now called the *Seiberg-Witten invariants*, obtained by counting

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solutions of a non-linear Dirac equation [**8, 10, 12, 22, 36**]. When a 4-manifold has a non-zero Seiberg-Witten invariant, a Weitzenböck argument shows that it cannot admit metrics of positive scalar curvature; and as a consequence, there are many simply connected, non-spin 4-manifolds which do not admit positive-scalar-curvature metrics. Since this last assertion stands in stark opposition to results concerning manifolds of higher dimension [**14, 34**], one can only conclude that dimension four must be treated as *sui generis*.

In fact, the idiosyncratic nature of four-dimensional geometry largely stems from a single Lie-group-theoretic fluke: *the four-dimensional rotation group $SO(4)$ isn't simple*. Indeed, the decomposition

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

induces an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{1}$$

of the bundle of 2-forms on any oriented Riemannian 4-manifold $(M, g)$. The rank-3 bundles $\Lambda^\pm$ are in fact exactly the eigenspaces of the Hodge (star) duality operator

$$\star : \Lambda^2 \longrightarrow \Lambda^2,$$

the eigenvalues of which are $\pm 1$; sections of $\Lambda^+$ are therefore called *self-dual* 2-forms, whereas sections of $\Lambda^-$ are called *anti-self-dual* 2-forms. Since $\star$ is unchanged on middle-dimensional forms if $g$ is multiplied by a smooth positive function, the decomposition (1) really only depends on the conformal class $\gamma = [g]$ rather than on the Riemannian metric $g$ itself.

Now this, in turn, has some peculiarly four-dimensional consequences for the Riemann curvature tensor $\mathcal{R}$. Indeed, since $\mathcal{R}$ may be identified with a linear map $\Lambda^2 \to \Lambda^2$, there is an induced decomposition [**32**]

$$\mathcal{R} = \left( \begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

into simpler pieces. Here the *self-dual* and *anti-self-dual Weyl curvatures* $W_\pm$ are defined to be the trace-free pieces of the appropriate blocks. The scalar curvature $s$ is understood to act by scalar multiplication, and $\mathring{r}$ can be identified with the trace-free part $r - \frac{s}{4}g$ of the Ricci curvature.

Witten's remarkable discoveries include the fact that the Seiberg-Witten equations (cf. §2 below) give a lower bound for the $\mathrm{L}^2$ norm of the scalar curvature [**23, 24, 39**]. As will be explained in this article, however, they also imply estimates [**26, 27**] which involve the $L^2$ norms of both $s$ and $W_+$. The importance of this is enhanced by the fact that the $L^2$ norms of the four pieces of the curvature tensor $\mathcal{R}$ are interrelated by two formulæ of Gauss-Bonnet type, so that the $L^2$ norms of $s$ and $W_+$ actually determine the $L^2$ norms of $W_-$ and $\mathring{r}$, too.

To clarify this last point, observe that the intersection form

$$\begin{array}{rcl} \smile : H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\phi]\ ,\ [\psi]\ ) & \longmapsto & \displaystyle\int_M \phi \wedge \psi \end{array}$$

may be diagonalized[(1)] as

$$\left[\begin{array}{cc} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{array}\right]$$

by choosing a suitable basis for the de Rham cohomology $H^2(M,\mathbb{R})$. The numbers $b_\pm(M)$ are independent of the choice of basis, and so are oriented homotopy invariants of $M$. Their difference

$$\tau(M) = b_+(M) - b_-(M),$$

is called the *signature* of $M$. The Hirzebruch signature theorem [**16**] asserts that this invariant is expressible as a curvature integral, which may be put in the explicit form [**32**]

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) d\mu. \tag{2}$$

Here the curvatures, norms $|\cdot|$, and volume form $d\mu$ are, of course, those of a particular Riemannian metric $g$, but the entire point is that the answer is independent of *which* metric we use. Thus the $L^2$ norms of $W_+$ and $W_-$ determine one another, once the signature $\tau$ is known.

A second such relation is given by the 4-dimensional case of the generalized Gauss-Bonnet theorem [**1**]. This asserts that the Euler characteristic

$$\chi(M) = 2 - 2b_1(M) + b_2(M)$$

(1) over $\mathbb{R}$, of course; the story over $\mathbb{Z}$ is a great deal more complicated!

is also given by a curvature integral, which can be put in the explicit form [**32**]

$$\chi(M)=\frac{1}{8\pi^2}\int_M\left(|W^+|^2+|W^-|^2+\frac{s^2}{24}-\frac{|\overset{\circ}{r}|^2}{2}\right)d\mu. \tag{3}$$

In conjunction with (2), this allows one to deduce the $L^2$ norm of $\overset{\circ}{r}$ from those of $s$ and $W_+$, assuming that $\chi$ and $\tau$ are both known.

For this reason, Seiberg-Witten theory is able to shed light on all four parts of the curvature tensor $\mathcal{R}$. In particular, we will see in §3 that these ideas naturally lead to subtle obstructions [**25, 19, 26, 27**] to the existence of Einstein metrics on 4-manifolds. Then, in §4, we will derive some new results regarding the optimality of the estimates of §2. It will turn out that this issue bears decisively on a conjecture regarding the existence of constant-scalar-curvature Kähler metrics on complex surfaces of general type.

## 2. Seiberg-Witten Theory

Let $(M,g)$ be a compact oriented Riemannian 4-manifold. On any contractible open subset $U\subset M$, one can define Hermitian vector bundles

$$\begin{array}{ccc}\mathbb{C}^2\to & \mathbb{S}_\pm|_U\\ & \downarrow\\ & U\subset M\end{array}$$

called spin bundles, with two characteristic properties: their determinant line bundles $\wedge^2\mathbb{S}_\pm$ are canonically trivial, and their projectivizations

$$\begin{array}{ccc}\mathbb{CP}_1\to & \mathbb{P}(\mathbb{S}_\pm)\\ & \downarrow\\ & M\end{array}$$

are exactly the unit 2-sphere bundles $S(\Lambda^\pm)$. As one passes between open subset $U$ and $U'$, however, the corresponding locally-defined spin bundles are not quite canonically isomorphic; instead, there are two equally 'canonical' isomorphisms, differing by a sign. Because of this, one cannot generally define the bundles $\mathbb{S}_\pm$ globally on $M$; manifolds on which this can be done are called *spin*, and are characterized by the vanishing of the Stiefel-Whitney class $w_2=w_2(TM)\in H^2(M,\mathbb{Z}_2)$. However, one can always find Hermitian complex line bundles $L\to M$ with first Chern class $c_1=c_1(L)$ satisfying

$$c_1\equiv w_2 \mod 2. \tag{4}$$

Given such a line bundle, one can then construct Hermitian vector bundles $\mathbb{V}_\pm$ with

$$\mathbb{P}(\mathbb{V}_\pm)=S(\Lambda^\pm)$$

by formally setting

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2},$$

because the sign problems encountered in consistently defining the transition functions of $\mathbb{S}_\pm$ are exactly canceled by those associated with trying to find consistent square-roots of the transition functions of $L$.

The isomorphism class $\mathfrak{c}$ of such a choice of $\mathbb{V}_\pm$ is called a *spin$^c$ structure* on $M$. The cohomology group $H^2(M,\mathbb{Z})$ acts freely and transitively on the spin$^c$ structures by tensoring $\mathbb{V}_\pm$ with complex line bundles. Each spin$^c$ structure has a first Chern class $c_1 := c_1(L) = c_1(\mathbb{V}_\pm) \in H^2(M,\mathbb{Z})$ satisfying (4), and the $H^2(M,\mathbb{Z})$-action on spin$^c$ structures induces the action

$$c_1 \longmapsto c_1 + 2\alpha,$$

$\alpha \in H^2(M,\mathbb{Z})$, on first Chern classes. Thus, if $H^2(M,\mathbb{Z})$ has trivial 2-torsion — as will be true, for example, if $M$ is simply connected — then the spin$^c$ structures are precisely in one-to-one correspondence with the set of cohomology classes $c_1 \in H^2(M,\mathbb{Z})$ satisfying (4).

To make this discussion more concrete, suppose that $M$ admits an almost-complex structure. Any given almost-complex structure can be deformed to an almost complex structure $J$ which is compatible with $g$ in the sense that $J^*g = g$. Choose such a $J$, and consider the rank-2 complex vector bundles

$$\begin{aligned} \mathbb{V}_+ &= \Lambda^{0,0} \oplus \Lambda^{0,2} \\ \mathbb{V}_- &= \Lambda^{0,1}. \end{aligned} \tag{5}$$

These are precisely the twisted spinor bundles of the spin$^c$ structure obtained by taking $L$ to be the anti-canonical line bundle $\Lambda^{0,2}$ of the almost-complex structure. A spin$^c$ structure $\mathfrak{c}$ arising in this way will be said to be of *almost-complex type*. These are exactly the spin$^c$ structures for which

$$c_1^2 = (2\chi + 3\tau)(M).$$

On a spin manifold, the spin bundles $\mathbb{S}_\pm$ carry natural connections induced by the Levi-Civita connection of the given Riemannian metric $g$. On a spin$^c$ manifold, however, there is not a natural unique choice of connections on $\mathbb{V}_\pm$. Nonetheless, since we formally have $\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2}$, every Hermitian connection $A$ on $L$ induces associated Hermitian connections $\nabla_A$ on $\mathbb{V}_\pm$.

On the other hand, there is a canonical isomorphism $\Lambda^1 \otimes \mathbb{C} = \mathrm{Hom}\,(\mathbb{S}_+, \mathbb{S}_-)$, so that $\Lambda^1 \otimes \mathbb{C} \cong \mathrm{Hom}\,(\mathbb{V}_+, \mathbb{V}_-)$ for any spin$^c$ structure, and this induces a canonical homomorphism

$$\mathit{Cliff}\ : \Lambda^1 \otimes \mathbb{V}_+ \longrightarrow \mathbb{V}_-$$

called *Clifford multiplication*. Composing these operations allows us to define a so-called *twisted Dirac operator*

$$D_A : \Gamma(\mathbb{V}_+) \longrightarrow \Gamma(\mathbb{V}_-)$$