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ANALYTIC CONTINUATION IN REPRESENTATION THEORY AND HARMONIC ANALYSIS

by

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Abstract. **—** In this paper we discuss topics in harmonic analysis and representation theory related to two different real forms G/H and G^c/H of a complex semisimple symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$. We connect representations of G and G^c using the theory of involutive representations of semi-groups and reflection symmetry. We discuss how to generalize the Segal-Bargmann transform to real forms of bounded symmetric domains. This transform maps $L^2(H/H\cap K)$ into the representation space of a highest weight representation of G. We show how this transform is related to reflection symmetry, which shows that it is a natural transform related to representation theory. Finally we discuss the connection of the H-spherical characters of the representations and relate them to spherical functions.

Résumé **(Prolongement analytique en théorie des représentations et analyse harmonique)**
Dans cet article, nous considérons des questions en analyse harmonique et en théorie des représentations concernant deux formes réelles différentes G/H et G^c/H d'un espace symétrique semi-simple complexe $G_{\mathbb{C}}/H_{\mathbb{C}}$. Nous établissons un lien entre les représentations de G et de G^c à l'aide de la théorie des représentations involutives des semi-groupes et la symétrie de réflexion. On examine la question de la généralisation de la transformée de Segal-Bargmann aux formes réelles des domaines symétriques bornés. Cette transformée envoie l'espace $L^c(H/H\cap K)$ dans l'espace de représentations d'une représentation du poids maximum de G. Nous montrons comment cette transformée est liée à la symétrie de réflexion, ce qui montre que c'est une transformée naturelle liée à la théorie des représentations. Finalement, on étudie la relation entre les caractères $H-$sphériques des représentations et les fonctions sphériques.

1. Introduction

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G_{\mathbb{C}}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We will for simplicity assume that $G\subset G_{\mathbb{C}}$ even if most of what we say holds also for the universal covering

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group $\tilde{G}$ and other connected groups locally isomorphic to G. Let $\theta : G \to G$ a Cartan involution on G and denote by $K = G^\theta$ the corresponding maximal compact subgroup of G. We will be interested in a special class of symmetric spaces, that are closely related to real forms of *bounded symmetric domains*. We will therefore assume that $D = G/K$ is a bounded symmetric domain. Let $\tau : D \to D$ by a *conjugation*, i.e., an anti-holomorphic involution, fixing the point $\{K\} \in D$. Those involutions were classified by A. Jaffee in [**25, 26**]. We will give the list later. We lift τ to an involution on G, $G_\mathbb{C}$, $\mathfrak{g}$, and $\mathfrak{g}_\mathbb{C}$. We will also denote those involutions by τ. Then τ commutes with θ. Let $H = G^\tau \subset H_\mathbb{C} := G_\mathbb{C}^\tau$. Then $D^\tau = H/H \cap K$.

On the Lie algebra level we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{k} \oplus \mathfrak{p} \\ &= (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})\,, \end{aligned}$$

$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$, $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau X = X\}$, and $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau X = -X\}$. Define a new real form of $\mathfrak{g}_\mathbb{C}$ by

$$\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$$

and let G^c be the corresponding real analytic subgroup of $G_\mathbb{C}$. Denote also by τ the restriction of τ to G^c. Let $H^c := G^{c\tau}$. Then $H = H^c = G \cap G^c$.

We have the following diagrams

$$\begin{array}{ccccc} & & M_\mathbb{C} \underset{\text{Complex}}{:=} G_\mathbb{C}/H_\mathbb{C} & & \\ & \nearrow & & \nwarrow & \\ M := G/H & & \text{Real forms} & & M^c := G^c/H \end{array}$$

and

$$D^\tau = H/H \cap K \underset{\text{Real form}}{\hookrightarrow} D = G/K\,.$$

The ideas that we discuss here are *how to analyze representations of* G, G^c, *and* H *via analytic continuation to open domains in* $M_\mathbb{C}$ *or by restriction to a real form.* The main tools are *involutive representations and positive definite kernels.* This can be expressed by the following simple diagram:

Unitary highest weight representations of G	$\longleftrightarrow$	Generalized principal series representations of G^c
$\updownarrow$	Reflection positivity H-spherical characters and spherical functions	$\updownarrow$
Restriction of holomorphic functions The Segal-Barmann transform		The subspace of functions with support in the open orbit HAN
$\searrow$		$\swarrow$
	Representations of H, Canonical representations	

Most of the ideas discussed here have been explained before in [**30, 31, 49, 51**]. We would in particular like to point to [**31**] for discussion on reflection positivity and highest weight representations. Several other people have been working on similar projects. We would like to point out here the following papers and preprints [**1, 2, 7, 8, 19, 41, 43, 56, 64, 68**].

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2. Unitary highest weight representations

We use the same notation and assumptions as in the introduction. In particular we assume that $G \subset G_{\mathbb{C}}$ is a Hermitian group, and $G_{\mathbb{C}}$ is simply connected. Thus $D = G/K$ is a bounded symmetric domain. The complex structure on D corresponds to an element $Z^0 \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(Z^0)$ has eigenvalues $0, i, -i$. The eigenspace corresponding to 0 is $\mathfrak{k}_\mathbb{C}$, and we denote by $\mathfrak{p}^+$, respectively $\mathfrak{p}^-$ the eigenspace corresponding to i respectively $-i$. Then both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras of $\mathfrak{g}_\mathbb{C}$ and

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^- .$$

Let $K_\mathbb{C} := \exp(\mathfrak{k}_\mathbb{C})$, and $P^\pm := \exp(\mathfrak{p}^\pm)$. The restriction of the exponential map is an isomorphism of $\mathfrak{p}^\pm$ onto $P^\pm$. The set $P^+K_\mathbb{C}P^-$ is open and dense in $G_\mathbb{C}$. Furthermore multiplication induces a diffeomorphism

$$P^+ \times K_\mathbb{C} \times P^- \ni (p,k,q) \mapsto pkq \in P^+K_\mathbb{C}P^- \subset G_\mathbb{C} .$$

We denote the inverse map by $x \mapsto (p(x), k(x), q(x))$. The Harish-Chandra bounded realization of D is given by

$$G/K \ni gK \mapsto \log(p(g)) \in \Omega_\mathbb{C} \subset \mathfrak{p}^+ \tag{2.1}$$

and $\Omega_\mathbb{C}$ is a bounded symmetric domain in $\mathfrak{p}^+$. Let $(\pi, \mathbf{H})$ be a representation of G in a Hausdorff, locally convex complete topological vector space $\mathbf{H}$, and let L be a closed subgroup of G. A vector $\mathbf{v} \in \mathbf{H}$ is called L-finite if $\pi(L)\mathbf{v}$ spans a finite dimensional subspace of $\mathbf{H}$. We call $\mathbf{v}$ smooth or analytic if for all $X \in \mathfrak{g}$ the map

$$\mathbb{R} \ni t \mapsto \pi(\exp(tX))\mathbf{v} \in \mathbf{H}$$

is smooth, or analytic respectively. We denote by $\mathbf{H}_L$, $\mathbf{H}^\infty$, $\mathbf{H}^\omega$ the space of L-finite, smooth, respectively analytic vectors in $\mathbf{H}$. Define a representation of $\mathfrak{g}$ on $\mathbf{H}^\infty$ by

$$d\pi(X)\mathbf{u} := \lim_{t\to 0} \frac{\pi(\exp tX)\mathbf{u} - \mathbf{u}}{t}, \quad \mathbf{u} \in \mathbf{H}^\infty .$$

We extend $d\pi$ by linearity to a representation of $\mathfrak{g}_\mathbb{C}$ and then of $U(\mathfrak{g})$, *the universal enveloping algebra of* $\mathfrak{g}$. The representations of G that we are mainly interested in are the *unitary highest weight representations* of G (see [**5, 6, 9, 10, 15, 16, 22, 27, 42, 61, 65, 67**] for further information.) Let $(\pi, \mathbf{H})$ be an admissible representation of G in a Banach space, and assume that the center of $U(\mathfrak{g})$ acts by scalars. Then $\mathbf{H}_K \subset \mathbf{H}^\omega$ and $\mathbf{H}_K$ is an $(U(\mathfrak{g}), K)$-module in the sense that it is both an $U(\mathfrak{g})$ and a K-module such that

$$X \cdot (k \cdot \mathbf{u}) = (\mathrm{Ad}(k)X) \cdot \mathbf{u}, \quad \forall k \in K, X \in \mathfrak{g}, \mathbf{u} \in \mathbf{H}_K .$$

We say that an $(U(\mathfrak{g}), K)$-module $(\pi, \mathbf{H})$ is *admissible* if the multiplicity of each irreducible representation of K in $\mathbf{H}$ is finite. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$ containing Z^0. Then $\mathfrak{t} \subset \mathfrak{z}_\mathfrak{g}(Z^0) \subset \mathfrak{k}$ so $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$.

Definition 2.1. — *Let* $\mathbf{H}$ *be an* $(U(\mathfrak{g}), K)$*-module. Then* π *is called a highest weight representation if there exists a Borel subalgebra* $\mathfrak{p} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{u} \subset \mathfrak{g}_\mathbb{C}$, $\lambda \in \mathfrak{t}_\mathbb{C}^*$, *and* $\mathbf{v} \in \mathbf{H}$ *such that the following holds:*

1. $X \cdot \mathbf{v} = \lambda(X)\mathbf{v}$ *for all* $X \in \mathfrak{t}$*;*
2. $\pi(\mathfrak{u})\mathbf{v} = 0$*;*
3. $U(\mathfrak{g})\mathbf{v} = \mathbf{H}$.

The element $\mathbf{v}$ *is called* primitive element of weight λ.

All the irreducible unitary highest weight representations of G can be constructed in a space of holomorphic functions on D for an appropriate choice of Z^0 or, which is the same, complex structure on D. For that let π be an irreducible representation of K. Let Δ denote the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. A root α is called *compact* if $\mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{k}_{\mathbb{C}}$. Otherwise α is called *non-compact*. Let Δ_c respectively Δ_n be the set of compact respectively non-compact roots. We choose the set Δ^+ of positive roots such that $\Delta_c^+ := \Delta^+ \cap \Delta_c$ is a system of positive roots for Δ_c and $\Delta_n^+ := \Delta^+ \cap \Delta_n = \{\alpha \in \Delta \mid \mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{p}^+\}$. Choose $H_\alpha \in i\mathfrak{t}$, such that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$. Let $W_K = W(\Delta_c) \subset W = W(\Delta)$ be the Weyl group generated by the reflections $s_\alpha(X) = X - \alpha(X)H_\alpha$ for $\alpha \in \Delta_c$ respectively $\alpha \in \Delta$. We denote the corresponding reflection on $i\mathfrak{t}^*$ by the same letter, i.e, $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. Then Δ_n^+ is invariant under W_K. Let $\sigma : G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the conjugation with respect to G. We sometimes write W or $\bar{g}$ for $\sigma(W)$, respectively $\sigma(g)$. We will usually use capital letters for the elements of the Lie algebra $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ except where we are viewing them as complex variables or elements in $\Omega_{\mathbb{C}}$.

Let $(\pi, \mathbf{V})$ be an irreducible representation of K with highest weight $\mu = \mu(\pi) \in i\mathfrak{t}^*$. For $z, v, w \in \mathfrak{p}^+$ and $g \in G_{\mathbb{C}}$ such that $g\exp(z), \exp(-w)\exp(v) \in P^+K_{\mathbb{C}}P^-$, let

$$g \cdot z := \log(p(g \exp z)) \tag{2.2}$$

$$J(g,z) := k(g \exp z)\,, \text{ and} \tag{2.3}$$

$$\kappa(v,w) := k(\exp(-\overline{w})\exp(v)). \tag{2.4}$$

Then the isomorphism in (2.1) intertwines the natural G-action on D with the action $(g, z) \mapsto g \cdot z$ of G on $\Omega_{\mathbb{C}}$. The function $J(g,z)$ is called *the universal factor of automorphy.* Finally we define

$$J_\pi(g,z) := \pi(J(g,z)) \quad \text{and} \quad K_\pi(z,w) := \pi(\kappa(z,w))^{-1}\,. \tag{2.5}$$

Then $z, w \mapsto K_\pi(z,w)$ is holomorphic in the first variable and anti-holomorphic in the second variable.

Let $S(\Omega_{\mathbb{C}}) := \left\{g \in G_{\mathbb{C}} \mid g^{-1} \cdot \Omega_{\mathbb{C}} \subset \Omega_{\mathbb{C}}\right\}$. Then $S(\Omega_{\mathbb{C}})$ is a maximal closed semigroup in $G_{\mathbb{C}}$ and there exists a maximal closed and convex G-invariant cone $W_{\max} \subset \mathfrak{g}$ such that $W_{\max} \cap -W_{\max} = \{0\}$ (pointed), $W_{\max} - W_{\max} = \mathfrak{g}$ (generating), and (see [**21, 23, 44**])

$$S(\Omega_{\mathbb{C}}) = G\exp(iW_{\max})\,.$$

Let $W_{\min} = \{X \in \mathfrak{g} \mid \forall Y \in W_{\max} : -B(X, \theta(Y)) \geq 0\}$, where B stands for the Killing form. Then $W_{\min}$ is a minimal G-invariant pointed and generating cone in $\mathfrak{g}$. Define $g^* = \sigma(g)^{-1}$. If $s = g\exp(iW) \in S(\Omega_{\mathbb{C}})$ then

$$s^* = \exp(iW)g^{-1} = g^{-1}\exp(i\mathrm{Ad}(g)W) \in S(\Omega_{\mathbb{C}})\,.$$

We notice the following well known and important lemma.