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GENERALIZED GRADIENTS AND POISSON TRANSFORMS

by

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Abstract. — For G a semisimple Lie group and P a parabolic subgroup we construct a large class of first-order differential operators which are G-equivariant between certain vector bundles over G/P. These are intertwining operators from one generalized principal series representation for G to another. We also study the relation with Poisson transforms to the Riemannian symmetric space G/K.

***Résumé* (Gradients généralisés et les transformations de Poisson).** — Pour G un groupe de Lie semi-simple et pour P un sous-groupe parabolique, nous construisons une grande famille d'opérateurs différentiels G-équivariants du premier ordre entre certains fibrés vectoriels sur G/P. Il s'agit d'opérateurs d'entrelacement des représentations de série principale généralisée. Nous étudions également la relation avec l'espace symétrique riemannien, G/K, en utilisant les transformations de Poisson.

1. Introduction

This paper is partly motivated by differential geometry, partly by representation theory for semi-simple Lie groups. We give a generalization of the results by Fegan [**4**], which dealt with the group $\mathrm{SO}(n,1)$, to the case of an arbitrary semisimple Lie group G and an arbitrary parabolic subgroup P. At the same time we give a new proof of Fegan's case, and place it in the framework of analysis on Lie groups. Our method of constructing intertwining first-order differential operators between generalized principal series representations for G has its origin in the generalized gradients of Stein and Weiss [**8**], suitably generalized to the setting of flag manifolds.

We expect our construction of these gradients to have applications in other parabolic geometries, and also in the construction of small unitary representations of semi-simple Lie groups. By duality our problem is related to finding embeddings between

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generalized Verma modules; this was studied for parabolic geometries in great generality by Cap, Slovák and Souček, see [**2**], and previously by Baston and Eastwood. Closest to our approach is the recent work by Korányi and Reimann [**6**] who with a different (and independent) method treat the case of a minimal parabolic subgroup. Note also [**10**] where a related family of operators is constructed and applied to the problem of finding composition series for real rank one groups.

Let us here briefly state in rough form our main result: Denote by $C^\infty(\mathbb{E})$ the smooth sections of a homogeneous vector bundle over the real flag manifold G/P, where G is a semi-simple Lie group, P a parabolic subgroup, and $\mathbb{E}$ induced by a representation E of P, i.e.

$$\mathbb{E} = G \times_P E.$$

We assume E irreducible, and denote by $\mathbb{T}^*$ the cotangent bundle over G/P with fiber T^* at the base point. The goal is to find a first-order differential operator on smooth sections

$$D\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{F})$$

which is G-equivariant between two such bundles. This is done by first finding an equivariant connection (actually in the first instance only equivariant w.r.t. the maximal compact subgroup K of G)

$$\nabla\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{E}\otimes\mathbb{T}^*)$$

and second to decompose the tensor product $E\otimes T^*$ and project on a suitable quotient F, invariant for the P-action:

$$\mathrm{proj}\colon E\otimes T^* \longrightarrow F.$$

Then our gradient is the composition $D = \mathrm{proj}\circ\nabla$ and we have

Theorem 1.1. — *In the setting above D is G-equivariant if and only if the Casimir operator of G has the same value in $C^\infty(\mathbb{E})$ and in $C^\infty(\mathbb{F})$.*

In the last section we show that these gradients can in some sense be extended to the Riemannian symmetric space G/K in a canonical way, which is consistent with natural vector-valued Poisson transforms from $C^\infty(\mathbb{E})$ to sections of bundles over G/K.

From a representation theory point of view the gradients D are useful in studying the lattice of invariant subspaces in $C^\infty(\mathbb{E})$, i.e. the composition series for generalized principal series. Though we shall not go into discussing higher order equivariant differential equations in this paper, it is clear that there will exist such by composing our first-order operators. The Poisson transforms relate to both representation theory and to geometric problems — we have at the end added one such example and also a case of a symplectic analogue on S^2 of the Dirac operator, equivariant for the double cover of the projective group.

2. Construction of gradients

Fix a semi-simple Lie group G with finite center, a maximal compact subgroup K, a corresponding Cartan decomposition of the Lie algebra of G:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

and a maximal Abelian subspace $\mathfrak{a}_0 \subseteq \mathfrak{s}$. We have a corresponding minimal parabolic subgroup $P_0 = M_0 A_0 N_0$ constructed in the usual way, and we fix a parabolic subgroup $P \supseteq P_0$ with Langlands decomposition

$$P = MAN$$

and for the Lie algebras $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ of M, A, N we get

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, θ the Cartan involution, and

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{n}_\alpha$$

the decomposition into the positive root spaces and $\alpha > 0$ means $\alpha \in \Delta^+ \subseteq \Delta$ for a choice of positive roots of $\mathfrak{a}$ in $\mathfrak{g}$. For this, see [**5**]. We shall also need the simple roots $S \subseteq \Delta^+$ (note that we can still talk about simple roots, even though we may not have a root system here). The flag manifold is the compact space

$$G/P = K/K \cap P = K/K \cap M$$

and this is where we shall construct equivariant first-order differential operators. Fix an irreducible finite-dimensional representation (σ, E_σ) of M in the Hilbert space E_σ (we shall later relax this condition); for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, the complex dual space to $\mathfrak{a}$, consider the generalized principal series representation

$$\pi_{\sigma,\nu} = \operatorname{Ind}_P^G(\sigma \otimes e^\nu \otimes I)$$

induced from the P-representation

$$(\sigma \otimes e^\nu \otimes I)(man) = \sigma(m)a^\nu.$$

The smooth vectors are

$$C^\infty(\mathbb{E}_{\sigma,\nu}) = \\ \big\{f\colon G \to E_\sigma \mid f \in C^\infty,\ f(gman) = \sigma(m)^{-1}a^{-\nu}f(g) \text{ for all } g \in G,\ man \in MAN\big\}$$

which we identify with the smooth sections of the homogeneous vector bundle

$$\mathbb{E}_{\sigma,\nu} = G \times_P E_{\sigma,\nu}$$

where $E_{\sigma,\nu} = E_\sigma$ with the P-action considered above. We call ν the weight of the representation/bundle.

As usual G acts by left translation:

$$(\pi_{\sigma,\nu}(g_0)f)(g) = f(g_0^{-1}g) \quad (g_0, g \in G)$$

and we may let this representation act in a Hilbert space by setting

$$\|f\|^2 = \int_K \|f(k)\|^2_{E_\sigma} dk$$

– but we shall not need to do so here. Recall that a first-order differential operator is a homomorphism from the first jet bundle to the image bundle, see [**7**], so we are looking for σ' and ν' with a

$$D\colon J^1(\mathbb{E}_{\sigma,\nu}) \longrightarrow \mathbb{E}_{\sigma',\nu'}$$

where the fiber at a point $x \in G/P$ of the first jet bundle is

$$J^1(\mathbb{E}_{\sigma,\nu})_x = C^\infty(\mathbb{E}_{\sigma,\nu})/Z^1_x(\mathbb{E}_{\sigma,\nu})$$

where

$$Z^1_x(\mathbb{E}_{\sigma,\nu}) = \{f \mid f^{(\alpha)}(x) = 0, \quad |\alpha| \leq 1\}$$

is the space of sections vanishing to first order at the point.

Now the first jet bundle is also a homogeneous vector bundle, and at the base point the fiber is (suppressing the σ and ν)

$$J^1(E) \cong E \oplus \mathrm{Hom}(\bar{\mathfrak{n}}, E)$$

and for a section $f \in C^\infty(\mathbb{E})$ we have the natural map

$$j^1\colon f \longrightarrow (f, df)|_{eP} \in J^1(E)$$

specifying for a section its value and first derivative at the base point. Here we identify the tangent space at the base point $T^* \cong \bar{\mathfrak{n}}$ and the cotangent space $T^* \cong \mathfrak{n}$ via the duality induced by the Killing form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. It is convenient to consider the derivative of a section as the following covariant derivative

$$(\nabla_X f)(g) = \frac{d}{dt} f(g \exp tX)|_{t=0} \quad (f \in C^\infty(\mathbb{E}), X \in \bar{\mathfrak{n}}, g \in G)$$

which defines a connection

$$\nabla\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{E} \otimes \mathbb{T}^*).$$

Our goal is to compose this with a projection from $E \otimes \mathfrak{n}$ onto some subspace F invariant under the action of M — this is the generalized gradient construction of the desired

$$D\colon C^\infty(\mathbb{E}) \longrightarrow C^\infty(\mathbb{F})$$

with an appropriate choice of weights. So we are looking for a G-map

$$D\colon G \times_P J^1(E) \longrightarrow G \times_P F$$

which means looking for a P-map

$$D\colon J^1(E) \longrightarrow F.$$

The main problem is to construct D as an $\mathfrak{n}$-map, i.e. we have to study the action of $\mathfrak{n}$ on the module $J^1(E)$. This is done in the following

Lemma 2.1. — *Let $v \in E$ and $A \in \mathrm{Hom}(\bar{\mathfrak{n}}, E)$ correspond to the section $f \in C^\infty(\mathbb{E})$ via the map j^1; then for all $Y \in \mathfrak{n}$ the action is*

$$Y \cdot (v, A) = (0, [Y, \cdot]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot v + A([Y, \cdot]_{\bar{\mathfrak{n}}}))$$

where an element $Z \in \mathfrak{g}$ is decomposed

$$Z = Z_{\bar{\mathfrak{n}}} + Z_{\mathfrak{m}\oplus\mathfrak{a}} + Z_{\mathfrak{n}}$$

according to the direct sum

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Proof. — Let $X \in \bar{\mathfrak{n}}$, $Y \in \mathfrak{n}$, $n = \exp sY$ and $f \in C^\infty(\mathbb{E})$, then the N-action on the differential of f is

$$\frac{d}{dt} f(n^{-1} \exp tX)|_{t=0} = \frac{d}{dt} f(\exp(\mathrm{Ad}(n^{-1})tX))|_{t=0}$$

and we also have by differentiation of this the action of Y as

$$\frac{d}{dt}\frac{d}{ds} \exp(\mathrm{Ad}(n^{-1})tX)|_{s=0}|_{t=0} = -\mathrm{ad}(Y)X = [X, Y]_{\bar{\mathfrak{n}}} + [X, Y]_{\mathfrak{m}\oplus\mathfrak{a}} + [X, Y]_{\mathfrak{n}}.$$

Since f is a section, it transforms trivially from the right under $\mathfrak{n}$ and according to the action in E under $\mathfrak{m} \oplus \mathfrak{a}$. Hence we get the $\mathfrak{n}$-action

$$[Y, X]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot f(e) + A([Y, X]_{\bar{\mathfrak{n}}})$$

as stated, since

$$A([Y, X]_{\bar{\mathfrak{n}}}) = [Y, X]_{\bar{\mathfrak{n}}} \cdot df(e).$$

□

Following Fegan we first consider the "$\mathfrak{m} \oplus \mathfrak{a}$" part in this action, namely the term

$$[Y, X]_{\mathfrak{m}\oplus\mathfrak{a}} \cdot v$$

which may be thought of as a map

$$\beta\colon \mathfrak{n} \longrightarrow \mathrm{Hom}(E, \mathfrak{n} \otimes E)$$

hence an element

$$\beta \in \mathrm{Hom}(\mathfrak{n}, E^* \otimes \mathfrak{n} \otimes E) \cong \mathrm{Hom}(\mathfrak{n} \otimes E, \mathfrak{n} \otimes E).$$

Now the image of β will be an $\mathfrak{n}$-submodule of $\mathfrak{n} \otimes E$ since β exactly encodes the action of $\mathfrak{n}$. The "$\bar{\mathfrak{n}}$" part is

$$A([X, Y]_{\bar{\mathfrak{n}}})$$

which can be made to vanish, namely by observing that if α is a simple root, then

$$\forall\, Y \in \mathfrak{n} \quad \forall\, X \in \bar{\mathfrak{n}}_\alpha \colon [X, Y]_{\bar{\mathfrak{n}}} = 0.$$

Hence for α a simple root, the image of

$$\beta\colon \mathfrak{n}_\alpha \otimes E \longrightarrow \mathfrak{n}_\alpha \otimes E$$