Séminaires & Congrès
4, 2000, p. 251–276

INVARIANT OPERATORS OF THE FIRST ORDER ON MANIFOLDS WITH A GIVEN PARABOLIC STRUCTURE

by

Jan Slovák & Vladimír Souček

Abstract. — The goal of this paper is to describe explicitly all invariant first order operators on manifolds equipped with parabolic geometries. Both the results and the methods present an essential generalization of Fegan's description of the first order invariant operators on conformal Riemannian manifolds. On the way to the results, we present a short survey on basic structures and properties of parabolic geometries, together with links to further literature.

Résumé **(Opérateurs invariants d'ordre 1 sur des variétés paraboliques).** — Le but de l'article est de décrire explicitement tous les opérateurs différentiels invariants d'ordre un sur les variétés munies d'une structure de géométrie parabolique (les espaces généralisés d'Élie Cartan). Les résultats, ainsi que les méthodes, généralisent un résultat de Fegan sur la classification des opérateurs différentiels d'ordre un sur une variété munie d'une structure conforme. Au passage, nous donnons un bref resumé des propriétés fondamentales des espaces généralisés d'É. Cartan et du calcul différentiel sur ces espaces.

1. Setting of the problem

Invariant operators appear in many areas of global analysis, geometry, mathematical physics, etc. Their analytical properties depend very much on the symmetry groups, which in turn determine the type of the background geometries of the underlying manifolds. The most appealing example is the so called conformal invariance of many distinguished operators like Dirac, twistor, and Yamabe operators in Riemannian geometry which lead to the study of all these operators in the framework of the natural bundles for conformal Riemannian geometries. Of course, mathematicians suggested a few schemes to classify all such operators and to discuss their properties from a universal point of view, usually consisting of a combination of geometric and

2000 ***Mathematics Subject Classification.*** — 53C15, 53A40, 53A30, 53A55, 53C05.
Key words and phrases. — Invariant operator, parabolic geometry, Casimir operator.

Supported by the GAČR, grant Nr. 201/99/0675.

algebraic tools. See e.g. [**41, 42, 43, 6, 7, 8, 33, 9, 28, 10**]. All of them combine, in different ways, ideas of representation theory of Lie algebras with differential geometry and global analysis.

In the context of problems in twistor theory and its various generalizations, the more general framework of representation theory of parabolic subgroups in semisimple Lie groups was suggested and links to the infinite dimensional representation theory were exploited, see e.g. the pioneering works [**4, 5**]. The close relation to the Tanaka's theory (cf. [**39, 40, 17, 44, 32, 13**]) was established and we may witness a fruitful interaction of all these ideas and the classical representation theory nowadays, see e.g. [**2, 3, 12, 14, 15, 16, 18, 22, 23, 24, 25**].

1.1. Parabolic geometries. — The name *parabolic geometry* was introduced in [**26**], following Fefferman's concept of *parabolic invariant theory*, cf. [**19, 20**], and it seems to be commonly adopted now. The general background for these geometries goes back to Klein's definition of geometry as the study of homogeneous spaces, which play the role of the flat models for geometries in the Cartan's point of view. Thus, following Cartan, the (curved) geometry in question on a manifold M is given by a first order object on a suitable bundle of frames, an absolute parallelism $\omega : T\mathcal{G} \to \mathfrak{g}$ for a suitable Lie algebra $\mathfrak{g}$ defined on a principal fiber bundle $\mathcal{G} \to M$ with structure group P whose Lie algebra is contained in $\mathfrak{g}$. On the Klein's homogeneous spaces themselves, there is the canonical choice — the left–invariant Maurer–Cartan form ω while on general $\mathcal{G}$, ω has to be equivariant with respect to the adjoint action and to recover the fundamental vector fields. These objects are called *Cartan connections* and they play the role of the Levi–Civita connections in Riemannian geometry in certain extent. A readable introduction to this background in a modern setting is to be found in [**35**]. The parabolic geometries, real or complex, are just those corresponding to the choices of parabolic subgroups in real or complex Lie groups, respectively.

Each linear representation $\mathbb{E}$ of the (parabolic) structure group P gives rise to the homogeneous vector bundle $E(G/P)$ over the corresponding homogeneous space G/P, and similarly there are the natural vector bundles $\mathcal{G} \times_P \mathbb{E}$ associated to each parabolic geometry on a manifold M. Analogously, more general natural bundles $\mathcal{G} \times_P \mathbb{S}$ are obtained from actions of P on manifolds $\mathbb{S}$.

Morphisms $\varphi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ are principal fiber bundle morphisms with the property $\varphi^*\omega' = \omega$. Obviously, the construction of the natural bundles is functorial and so we obtain the well defined action of morphisms of parabolic geometries on the sheaves of local sections of natural bundles. In particular, the *invariant operators* on manifolds with parabolic geometries are then defined as those operators on such sections commuting with the above actions.

1.2. First order linear operators. — In this paper, first order linear differential operators between natural vector bundles $E(M)$, $E'(M)$ are just those differential operators which are given by linear morphisms $J^1E(M) \to E'(M)$. For example, for conformal Riemannian geometries this means that the (conformal) metrics may enter in any differential order in their definition.

The mere existence of the absolute parallelism ω among the defining data for a parabolic geometry on M yields an identification of all first jet prolongations J^1EM of natural bundles with natural bundles $\mathcal{G} \times_P J^1\mathbb{E}$ for suitable representations $J^1\mathbb{E}$ of P, see 2.4 below. Moreover, there is the well known general relation between invariant differential operators on homogeneous vector bundles and the intertwining morphisms between the corresponding jet modules. Thus, we see immediately that each first order invariant operator between homogeneous vector bundles over G/P extends canonically to the whole category of parabolic geometries of type (G, P). We may say that they are given explicitly by their symbols (which are visible on the flat model G/P) and by the defining Cartan connections ω.

On the other hand, the invariants of the geometries may enter into the expressions of the invariant operators, i.e. we should consider also all possible contributions from the curvature of the Cartan connection ω. This leads either to operators which are not visible at all on the (locally) flat models, or to those which share the symbols with the above ones and again the difference cannot be seen on the flat models.

In this paper we shall not deal with such curvature contributions. In fact, we classify all invariant first order operators between the homogeneous bundles over the flat models, which is a purely algebraic question. In the above mentioned sense, they all extend canonically to all curved geometries.

At the same time, there are strict analogies to the Weyl connections from conformal Riemannian geometries available for all parabolic geometries and so we shall also be able to provide explicit universal formulae for all such operators from the classification list in terms of these linear connections on the underlying manifolds.

This was exactly the output of Fegan's approach in the special case of $G = \mathrm{SO}(m+1, 1)$, P the Poincaré conformal group, which corresponds to the conformal Riemannian geometries, [**21**]. Since the conformal Riemannian geometries are uniformly one–flat (i.e. the canonical torsion vanishes), this also implies that all first order operators on (curved) conformal manifolds, which depend on the conformal metrics up to the first order, are uniquely given by their restrictions to the flat conformal spheres. We recover and vastly extend his approach. In particular, we prove the complete algebraic classification for all parabolic subgroups in semisimple Lie groups G. Moreover, rephrasing the first order dependence on the structure itself by the assumption on the homogeneity of the operator, we obtain the unique extension of our operators for all parabolic geometries with vanishing part of torsion of homogeneity one.

We also show that compared to the complexity of the so called standard operators of all orders in the Bernstein–Gelfand–Gelfand sequences, constructed first in [**16**] and developed much further in [**11**], the original Fegan's approach to first order operators is surprisingly powerful in the most general context.

Although the algebraic classification of the invariant operators does not rely on the next section devoted to a survey on general parabolic geometries, we prefer to include a complete line of arguments leading to full understanding of the curved extensions of the operators and their explicit formulae in terms of the underlying Weyl connections.

2. Parabolic geometries, Weyl connections, and jet modules

2.1. Regular infinitesimal flag structures. — The homogeneous models for parabolic geometries are the (real or complex) generalized flag manifolds G/P with G semisimple, P parabolic. It is well known that on the level of the Lie algebras, the choice of such a pair $(\mathfrak{g}, \mathfrak{p})$ is equivalent to a choice of the so called $|k|$–grading of a semisimple $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$
$$\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$
$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \simeq \mathfrak{g}/\mathfrak{p}.$$

Then the Cartan–Killing form provides the identification $\mathfrak{g}_i^* = \mathfrak{g}_{-i}$ and there is the Hodge theory on the cohomology $H^*(\mathfrak{g}_-, \mathbb{W})$ for any $\mathfrak{g}$–module $\mathbb{W}$, cf. [**40, 44, 13, 16**].

Now, the Maurer–Cartan form ω distributes these gradings to all frames $u \in G$ and all P–equivariant data are projected down to the flag manifolds G/P. This construction goes through for each Cartan connection of type (G, P) and so there is the filtration

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M \tag{1}$$

on the tangent bundle TM of each manifold M underlying the principal fiber bundle $\mathcal{G} \to M$ with Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, induced by the inverse images of the P–invariant filtration of $\mathfrak{g}$. Moreover, the same absolute parallelism ω induces the reduction of the structure group of the associated graded tangent bundle

$$\operatorname{Gr} TM = (T^{-k}M/T^{-k+1}) \oplus \cdots \oplus (T^{-2}M/T^{-1}M) \oplus T^{-1}M$$

to the reductive part G_0 of P. In particular, this reduction introduces an algebraic bracket on $\operatorname{Gr} TM$ which is the transfer of the G_0–equivariant Lie bracket in $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$.

Next, let M be any manifold, $\dim M = \dim \mathfrak{g}_-$. An *infinitesimal flag structure of type* (G, P) on M is given by a filtration (1) on TM together with the reduction of the associated graded tangent bundle to the structure group G_0 of the form $\operatorname{Gr} T_xM \simeq \operatorname{Gr} \mathfrak{g}_-$, with the freedom in G_0, at each $x \in M$.

Let us write $\{\ ,\ \}_{\mathfrak{g}_0}$ for the induced algebraic bracket on $\operatorname{Gr} TM$. The infinitesimal flag structure is called *regular* if $[T^iM, T^jM] \subset T^{i+j}M$ for all $i, j < 0$ and the algebraic bracket $\{\ ,\ \}_{\mathrm{Lie}}$ on $\mathrm{Gr}\, TM$ induced by the Lie brackets of vector fields on M coincides with $\{\ ,\ \}_{\mathfrak{g}_0}$. It is not difficult to observe that the infinitesimal structures underlying Cartan connections ω are regular if and only if there are only positive homogeneous components of the curvature κ of ω, cf. [**34, 14**].

The remarkable conclusion resulting from the general theory claims that for each regular infinitesimal flag structure of type (G, P) on M, under suitable normalization of the curvature κ (its co–closedness), there is a unique Cartan bundle $\mathcal{G} \to M$ and a unique Cartan connection ω on $\mathcal{G}$ of type (G, P) which induces the given infinitesimal flag structure, up to isomorphisms of parabolic geometries and with a few exceptions, see [**40, 32, 13**] or [**14**], sections 2.7–2.11., for more details.

2.2. Examples. — The simplest and best known situation occurs for $|1|$–graded algebras, i.e. $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then the filtration is trivial, $TM = T^{-1}M$, and the regular infinitesimal flag structures coincide with standard G_0–structures, i.e. reductions of the structure group of TM to G_0. The examples include the conformal, almost Grassmannian, and almost quaternionic structures. The projective structures correspond to $\mathfrak{g} = \mathfrak{sl}(m+1, \mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{gl}(m, \mathbb{R})$, and this is one of the exceptions where some more structure has to be chosen in order to construct the canonical Cartan connection ω. The series of papers [**15**] is devoted to all these geometries.

Next, the $|2|$–graded examples include the so called parabolic contact geometries and, in particular, the hypersurface type non–degenerate CR-structures. See e.g. [**44, 14**] for more detailed discussions. Further examples of geometries are given by the Borel subalgebras in semisimple Lie algebras, and they are modeled on the full flag manifolds G/P.

2.3. The invariant differential. — The Cartan connection ω defines the *constant vector fields* $\omega^{-1}(X)$ on $\mathcal{G}$, $X \in \mathfrak{g}$. They are defined by $\omega(\omega^{-1}(X)(u)) = X$, for all $u \in \mathcal{G}$. In particular, $\omega^{-1}(Z)$ is the fundamental vector field if $Z \in \mathfrak{p}$. The constant fields $\omega^{-1}(X)$ with $X \in \mathfrak{g}_-$ are called *horizontal*.

Now, let us consider any natural vector bundle $EM = \mathcal{G} \times_P \mathbb{E}$. Its sections may be viewed as P–equivariant functions $s : \mathcal{G} \to \mathbb{E}$ and the Lie derivative of functions with respect to the constant horizontal vector fields defines the *invariant derivative* (with respect to ω)

$$\nabla^\omega : C^\infty(\mathcal{G}, \mathbb{E}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{E})$$
$$\nabla^\omega s(u)(X) = \mathcal{L}_{\omega^{-1}(X)} s(u).$$

We also write $\nabla^\omega_X s$ for values with the fixed argument $X \in \mathfrak{g}_-$.

The invariant differentiation is a helpful substitute for the Levi–Civita connections in Riemannian geometry, but it has an unpleasant drawback: it does not produce P–equivariant functions even if restricted to equivariant $s \in C^\infty(\mathcal{G}, \mathbb{E})^P$. One possibility how to deal with that is to extend the derivative to all constant fields, i.e. to consider