Séminaires & Congrès
6, 2002, p. 1–41

UPDATE ON TORIC GEOMETRY

by

David A. Cox

Abstract. — This paper will survey some recent work on toric varieties. The goal is to help the reader understand how the papers in this volume relate to current trends in toric geometry.

Introduction

In recent years, toric varieties have been an active area of research in algebraic geometry. This article will give a partial overview of the work on toric geometry done since the 1995 survey paper [**90**]. One of our main goals is to help the reader understand the larger context of the eight papers in this volume:

[**74**] *Semigroup algebras and discrete geometry* by W. Bruns and J. Gubeladze.
[**93**] *How to calculate A-Hilb* $\mathbb{C}^3$ by A. Craw and M. Reid.
[**94**] *Crepant resolutions of Gorenstein toric singularities and upper bound theorem* by D. Dais.
[**96**] *Resolving* 3*-dimensional toric singularities* by D. Dais.
[**140**] *Producing good quotients by embedding into a toric variety* by J. Hausen.
[**159**] *Special McKay correspondence* by Y. Ito.
[**230**] *Lectures on height zeta functions of toric varieties* by Y. Tschinkel.
[**234**] *Toric Mori theory and Fano manifolds* by J. Wiśniewski.

These papers (and many others) were presented at the 2000 Summer School on the Geometry of Toric Varieties held at the Fourier Institute in Grenoble.

We will assume that the reader is familiar with basic facts about toric varieties. We will work over an algebraically closed field k and follow the notation used in Fulton [**121**] and Oda [**196**], except that we use Σ to denote a fan. Recall that one can

2000 ***Mathematics Subject Classification.*** — 14M25.
Key words and phrases. — Toric varieties.

think of a toric variety in many ways. First, we have the union of affine toric varieties presented by Fulton [**121**] and Oda [**196**]:

$$X_\Sigma = \bigcup_{\sigma\in\Sigma} X_\sigma, \quad X_\sigma = \mathrm{Spec}(k[\sigma^\vee \cap M]). \tag{0.1}$$

Second, when the support of Σ spans $N_\mathbb{R}$, we have the categorical quotient representation considered by Cox [**89**]:

$$X_\Sigma = \big(k^{\Sigma(1)} \smallsetminus \mathbf{V}(B)\big)/G, \quad G = \mathrm{Hom}(A_{n-1}(X_\Sigma), k^*), \tag{0.2}$$

where $B = \langle x^{\hat\sigma} : \sigma \in \Sigma\rangle$ and $x^{\hat\sigma} = \prod_{\rho\notin\sigma(1)} x_\rho$. We call $S = k[x_\rho : \rho \in \Sigma(1)]$ the *homogeneous coordinate ring* of X_Σ, which is graded by $A_{n-1}(X_\Sigma)$. The representation (0.2) is a geometric quotient if and only if Σ is simplicial.

Finally, $\mathcal{A} = \{m_1,\dots,m_\ell\} \subset \mathbb{Z}^n$ gives the semigroup algebra $k[t^{m_1},\dots,t^{m_\ell}] \subset k[t_1^{\pm1},\dots,t_n^{\pm1}]$. Then we have the (possibly non-normal) affine toric variety discussed by Sturmfels [**223, 224**]:

$$X_\mathcal{A} = \mathrm{Spec}(k[t^{m_1},\dots,t^{m_\ell}]). \tag{0.3}$$

The map $x_i \mapsto t^{m_i}$ gives a surjection $k[x_1,\dots,x_\ell] \to k[t^{m_1},\dots,t^{m_\ell}]$ whose kernel

$$I_\mathcal{A} = \ker(k[x_1,\dots,x_\ell] \to k[t^{m_1},\dots,t^{m_\ell}]) \tag{0.4}$$

is the toric ideal of $\mathcal{A}$. This ideal is generated by binomials and is the defining ideal of $X_\mathcal{A} \subset k^\ell$. If $I_\mathcal{A}$ is homogeneous, then $X_\mathcal{A}$ is the affine cone over the (possibly non-normal) projective toric variety $Y_\mathcal{A} \subset \mathbb{P}^{\ell-1}$.

This survey concentrates on work done since our earlier survey [**90**]. Hence most of the papers we discuss appeared in 1996 or later. We caution the reader in advance that our survey is not complete, partly for lack of space and partly for ignorance on our part. We apologize for the many fine papers not mentioned below.

1. The Minimal Model Program and Fano Toric Varieties

The paper [**234**] by Jarosław Wiśniewski discusses toric Mori theory and Fano varieties. The main goal of the paper is to illustrate aspects of the minimal model program using toric varieties. As Wiśniewski points out, toric varieties are rational and hence trivial from the point of view of the minimal model program. Nevertheless, many hard results about minimal models can be proved without difficulty in the toric case. It makes for an excellent introduction to the subject.

An important feature of the minimal model program is that singularities are unavoidable in higher dimensions. In our discussion of Wiśniewski's lectures, we will assume that X is a normal projective variety such that K_X is $\mathbb{Q}$-Cartier (meaning that some positive integer multiple of K_X is a Cartier divisor). Such a variety is called $\mathbb{Q}$-*Gorenstein*. Given a resolution of singularities $\pi : Y \to X$, we can write

$$K_Y = \pi^*(K_X) + \textstyle\sum_i d_i E_i$$

where the exceptional set $E = \bigcup_i E_i$ is a divisor with normal crossings. We call $\sum_i d_i E_i$ the *discrepancy divisor*. Then we say that the singularities X are:

$$\begin{aligned} &\textit{terminal} \text{ if } d_i > 0 \text{ for all } i;\\ &\textit{canonical} \text{ if } d_i \geqslant 0 \text{ for all } i;\\ &\textit{log-terminal} \text{ if } d_i > -1 \text{ for all } i; \text{ and}\\ &\textit{log-canonical} \text{ if } d_i \geqslant -1 \text{ for all } i. \end{aligned} \tag{1.1}$$

Furthermore, $\pi : Y \to X$ is *crepant* if the discrepancy is zero, i.e., $d_i = 0$ for all i or, equivalently, $K_Y = \pi^*(K_X)$. In Section 2.2, we will explain what these singularities mean in the toric case.

1.1. Extremal Rays, Contractions, and Flips. — The first three lectures in Wiśniewski's article [**234**] are based primarily on Reid [**209**] and discuss aspects of the minimal model program related to the Mori cone $NE(X)$, which is the cone of $H_2(X, \mathbb{R})$ generated by homology classes of irreducible curves on X. For a simplicial toric variety, $NE(X)$ is generated by the torus-invariant curves in X (which correspond to codimension 1 cones of the fan of X). In [**234**, Lec. 1], Wiśniewski describes in detail how this relates to Mori's move-bend-break strategy.

When X is projective, the 1-dimensional faces of $NE(X)$ are *extremal rays*. In the toric case, it follows that each extremal ray is the class of a torus-invariant curve in X. Wiśniewski contrasts this with the Cone Theorem of Mori and Kawamata, which for a general variety X gives only a partial description of $NE(X)$.

Extremal rays are important in the minimal model program because of the Contraction Theorem of Kawamata and Shokurov, which asserts that if a projective variety X has terminal singularities, then every Mori ray R (= an extremal ray with $R \cdot K_X < 0$) gives a contraction

$$\varphi_R : X \longrightarrow X_R$$

with connected fibers such that X_R is normal and projective and a curve in X is contracted to a point if and only if its class lies in R.

For an extremal ray R on a simplicial projective toric variety of dimension n, Wiśniewski gives Reid's construction [**209**] of the corresponding contraction. Here is a brief summary. Given R, define α and β to be

$$\begin{aligned} \alpha &= |\{D_\rho : D_\rho \cdot R < 0\}| \\ \beta &= n + 1 - |\{D_\rho : D_\rho \cdot R > 0\}|, \end{aligned}$$

where the D_ρ are the torus-invariant divisors of X. These will be important invariants of the contraction φ_R. The formulas given in [**234**, Lec. 2] show that α and β are easy to compute in practice.

Now let ω be a codimension 1 cone in the fan Σ of X such that the corresponding curve lies in R. Then ω is a face of two top-dimensional cones δ, δ' in Σ. One can show that the sum $\delta + \delta'$ is again a convex cone. Then consider the "fan" Σ_R^* obtained from Σ by removing all such ω's and, for each such ω, replacing the corresponding δ, δ' with $\delta + \delta'$. We put "fan" in parenthesis because of the following result.

Lemma 1.1. — *If $\alpha > 0$, then Σ_R^* is a fan, but if $\alpha = 0$, then there is a subspace $\mu(R)$ of dimension $n - \beta$ such that $\sigma \cap -\sigma = \mu(R)$ for every cone $\sigma \in \Sigma_R^*$.*

The extremal ray R then gives the desired contraction $\varphi_R : X \to X_R$ as follows:

– When $\alpha = 0$, Σ_R^* is a *degenerate fan.* Then $\Sigma_R^*/\mu(R)$ becomes a fan in $N_{\mathbb{R}}/\mu(R)$. Furthermore, if X_R is the toric variety of $\Sigma_R^*/\mu(R)$, then X_R has dimension β and φ_R is a toric fibration whose fibers are weighted projective spaces.
– When $\alpha > 0$, then Σ_R^* is a fan, and if X_R is the toric variety of Σ_R^*, then φ_R is birational. Furthermore:
  - If $\alpha = 1$, then φ_R is the blow-up of a subset of X_R of dimension $\beta - 1$. Thus the exceptional set is a divisor. Also, X_R is terminal if X is.
  - If $\alpha > 1$, then the exceptional set of φ_R has codimension > 1, so that φ_R is an isomorphism in codimension 2. We say that R is a *small ray.*

Notice how degenerate fans arise naturally in this context.

In terms of the minimal model program, the cases when $\alpha = 0$ or 1 work nicely, since in these cases we can replace X with X_R. But $\alpha > 1$ causes problems because in this case, the cones $\delta + \delta'$ are not simplicial, so that X_R has bad singularities from the minimal model point of view. This is where the next big result of the minimal model program comes into play, the Flip Theorem. This is more properly called the Flip Conjecture, since for general varieties, it has been proved only for dimension $\leqslant 3$ (by Mori). However, it is true for all dimensions in the toric case.

The rough idea is that when R is a small ray, X_R isn't suitable, so instead we "flip" R to $-R$ on a birational model X_1 and then replace X with X_1. More precisely, the Toric Flip Theorem, as stated in [**234**, Lec. 3], constructs a fan Σ_1 with toric variety X_1 and a birational map

$$\psi : X_1 \dashrightarrow X$$

with the following properties:

– If X is terminal with $K_X \cdot R < 0$ (i.e., R is a Mori ray), then X_1 is terminal.
– ψ is an isomorphism in codimension 1.
– $R_1 = -\psi^*(R)$ is an extremal ray for X_1 and $\varphi_1 = \varphi_R \circ \psi : X_1 \to X_R$ is the corresponding contraction of R_1.

Furthermore, Σ_1 is easy to construct: using the natural decomposition of $\delta + \delta'$ into simplices described in [**234**, Lec. 3], one simply replaces each cone $\delta + \delta' \in \Sigma_R^*$ with these simplices.

There are some recent papers related to these topics. First, concerning extremal rays, Bonavero [**47**] observes that if X is a projective toric variety and $\pi : X \to X'$ is a smooth toric blow-down, then X' is projective if and only if a line contained in a non-trivial fiber of π is an extremal ray. He then uses this to classify certain smooth blow-downs to non-projective varieties. Second, concerning minimal models, if $Y \subset X$ is a hypersurface in a complete toric variety such that the intersection of Y with every orbit is either empty or transverse of codimension 1, then S. Ishii [**157**] uses the toric framework described above to show that minimal model program works for Y, as described in the introduction to [**234**]. See also Ishii's paper [**156**].

Returning to the lectures [**234**], Wiśniewski points out that when X is toric and projective, any face of $NE(X)$ can be contracted, not just edges (= extremal rays). This is not true for general projective varieties. Then [**234**, Lec. 3] ends with a discussion of toric flips from the point of view of Morelli-Włodarczyk corbodisms, which is based on the work of Morelli [**189**] and Włodarczyk [**236**]. In [**234**, Lec. 4], Wiśniewski defines terminal and canonical singularities as in (1.1) and explains how these relate to the toric versions of the Contraction Theorem and Flip Theorem. He also describes the Euler sequence of a smooth toric variety.

1.2. Fano Varieties. — In [**234**, Lec. 5], Wiśniewski discusses Fano varieties. In general, a normal variety X is Fano when some multiple of $-K_X$ is an ample Cartier divisor. As explained in the introduction to [**234**], part of the minimal model program includes Fano-Mori fibrations, whose fibers are Fano varieties. Wiśniewski focuses on the case of toric Fano manifolds for simplicity.

Results of Batyrev show that in any given dimension, there are at most finitely many toric Fano manifolds (up to isomorphism). In dimension 2, it is easy to see that there are only five: $\mathbb{P}^1 \times \mathbb{P}^1$ together with the blow-up of $\mathbb{P}^2$ at $0, 1, 2$ or 3 fixed points of the torus action. In dimension 3, Wiśniewski sketches the proof that there are precisely 18 smooth toric Fano 3-folds. He also discusses the classification of non-toric Fano manifolds, where the situation is considerably more complicated.

In dimension 4, Batyrev [**28**] recently published a classification of smooth toric Fano 4-folds. As noted by Sato [**220**], Batyrev missed one, so that Batyrev's list of 123 is now a list of 124 smooth toric Fano 4-folds. The key point is that toric Fano manifolds of dimension n correspond to n-dimensional lattice polytopes $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ with the origin as an interior point such that the vertices of every facet are a basis of N. (Given such a P, the cones over the faces of P give a fan whose toric variety is a Fano manifold.) Hence the proof reduces to classifying the possible polytopes.

One can generalize the polytopes of the previous paragraph to the idea of a *Fano polytope*. This is an n-dimensional lattice polytope $P \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ with the property that 0 is the unique lattice point in the interior of P. In this case, taking cones over faces as above gives a Fano toric variety X. Furthermore, the singularities of S can be read off from the polytope. For example, Section 2.2 below implies that:

- If the only lattice points in P are 0 plus the vertices, then X has terminal singularities.
- If every facet of P is defined by an equation of the form $\langle m, u\rangle = 1$ for some $m \in M$, then X is Gorenstein.