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HOW TO CALCULATE A-Hilb $\mathbb{C}^3$

by

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Abstract. — Nakamura [Iku Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom. 10 (2001), no. 4, 757–779] introduced the G-Hilbert scheme G-Hilb $\mathbb{C}^3$ for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, and conjectured that it is a crepant resolution of the quotient $\mathbb{C}^3/G$. He proved this for a diagonal Abelian group A by introducing an explicit algorithm that calculates A-Hilb $\mathbb{C}^3$. This note calculates A-Hilb $\mathbb{C}^3$ much more simply, in terms of fun with continued fractions plus regular tesselations by equilateral triangles.

1. Statement of the result

1.1. The junior simplex and three Newton polygons. — Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be a diagonal subgroup acting on $\mathbb{C}^3$. Write $L \supset \mathbb{Z}^3$ for the overlattice generated by all the elements of A written in the form $\frac{1}{r}(a_1, a_2, a_3)$. The junior simplex Δ (compare [**IR**], [**R**]) has 3 vertexes

$$e_1 = (1,0,0), \quad e_2 = (0,1,0) \quad \text{and} \quad e_3 = (0,0,1).$$

Write $\mathbb{R}^2_\Delta$ for the affine plane spanned by Δ, and $\mathbb{Z}^2_\Delta = L \cap \mathbb{R}^2_\Delta$ for the corresponding affine lattice. Taking each e_i in turn as origin, construct the Newton polygons obtained as the convex hull of the lattice points in $\Delta \smallsetminus e_i$ (see Figure 1.a):

$$f_{i,0}, f_{i,1}, f_{i,2}, \dots, f_{i,k_i+1}, \tag{1.1}$$

where $f_{i,0}$ is the primitive vector along the side $[e_i, e_{i-1}]$, and f_{i,k_i+1} that along $[e_i, e_{i+1}]$. (The indices $i, i\pm1$ are cyclic. Also, since e_i is the origin, the notation $f_{i,j}$ denotes both the lattice point of Δ and the corresponding vector $e_i f_{i,j}$.) The vectors $f_{i,j}$ out of e_i are subject to the Jung–Hirzebruch continued fraction rule:

$$f_{i,j-1} + f_{i,j+1} = a_{i,j} \cdot f_{i,j} \quad \text{for } j = 1, \dots, k_i, \tag{1.2}$$

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where $a_{i,j} \geqslant 2$. Here $r_i/\alpha_i = [a_{i,1}, \dots, a_{i,k_i}]$ comes from expressing $\mathbb{Z}^2_\Delta$ in terms of the cone at e_i, writing

$$\mathbb{Z}^2_\Delta = \mathbb{Z}^2(f_{i,0}, f_{i,k_i+1}) + \mathbb{Z} \cdot f_{i,1} = \mathbb{Z}^2 + \mathbb{Z} \cdot \tfrac{1}{r_i}(\alpha_i, 1),$$

with $\alpha_i < r$ and coprime to r. Write L_{ij} for the line out of e_i extending or equal to the initial segment $[e_i, f_{ij}]$ (*line* is *line segment* throughout). The resulting fan at e_i corresponds to the Jung–Hirzebruch resolution of the surface singularity $\mathbb{C}^2_{(x_i=0)}/A$. The picture so far is the simplex Δ with a number of lines L_{ij} growing out of each of the 3 vertexes (Figure 1.a).

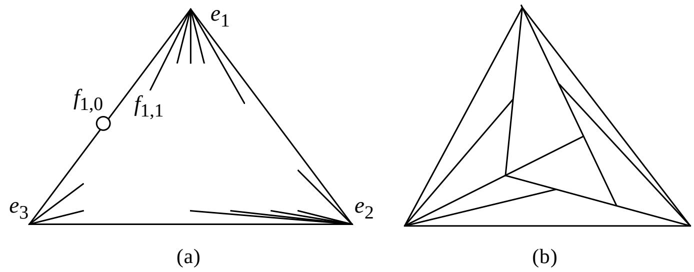

FIGURE 1. (a) Three Newton polygons; (b) subdivision into regular triangles

1.2. Regular triangles. — Write $\mathbb{Z}^2$ for the group of translations of the affine lattice $\mathbb{Z}^2_\Delta$. A *regular triple* is a set of three vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$, any two of which form a basis of $\mathbb{Z}^2$, and such that $\pm v_1 \pm v_2 \pm v_3 = 0$. (The standard regular triple is $\pm(1,0), \pm(0,1), \pm(1,1)$; it appears all over elementary toric geometry, for example, as the fan of $\mathbb{P}^2$ or the blowup of $\mathbb{A}^2$.) We are only concerned with regular triples among the vectors $f_{i,j}$ introduced in 1.1.

As usual, a *lattice triangle* T is a triangle $T \subset \mathbb{R}^2_\Delta$ with vertexes in $\mathbb{Z}^2_\Delta$. We say that T is a *regular triangle* if each of its sides is a line L_{ij} extending some $[e_i, f_{i,j}]$ and the 3 primitive vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$ pointing along its sides form a regular triple.

It is easy to see that a regular triangle T is affine equivalent to the triangle with vertexes $(0,0), (r,0), (0,r)$ for some $r \geqslant 1$, called the *side* of T. Its *regular tesselation* is that shown in Figure 2.a: a regular triangle of side r subdivides into r^2 basic triangles with sides parallel to v_1, v_2, v_3.

A regular triangle is the thing you get as the junior simplex for the group

$$A = \mathbb{Z}/r \oplus \mathbb{Z}/r = \left\langle \frac{1}{r}(1,-1,0), \frac{1}{r}(0,1,-1), \frac{1}{r}(-1,0,1) \right\rangle \subset \mathrm{SL}(3,\mathbb{C})$$

(the maximal diagonal subgroup of exponent r). The tesselation consists of basic triangles with vertexes in Δ, so corresponds to a crepant resolution of the quotient singularity. It is known (see 3.2 below and [**R**], Example 2.2) that in this case A-Hilb $\mathbb{C}^3$ is the toric variety associated with its regular tesselation.

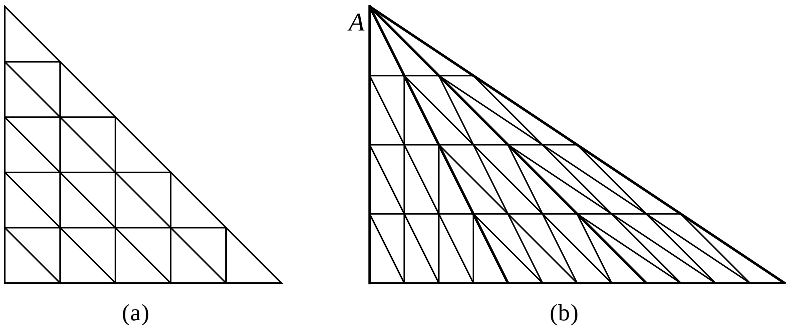

FIGURE 2. (a) A 5-regular triangle; (b) a $(4,12)$-semiregular triangle (see 2.8.3)

1.3. The main result

Theorem 1.1. — *The regular triangles partition the junior simplex* Δ.

Section 2 gives an easy continued fraction procedure determining the partition; Figure 1.b illustrates the rough idea, and worked examples are given in 2.6 below[(1)] (see Figures 6–8).

Theorem 1.2. — *Let* Σ *denote the toric fan determined by the regular tesselation (see 1.2) of all regular triangles in the junior simplex* Δ*. The associated toric variety* Y_Σ *is Nakamura's A-Hilbert scheme* A-Hilb $\mathbb{C}^3$.

***Corollary 1.3* (Nakamura)**. — A-Hilb $\mathbb{C}^3 \to \mathbb{C}^3/A$ *is a crepant resolution.*

Corollary 1.4. — *Every compact exceptional surface in* A-Hilb $\mathbb{C}^3$ *is either* $\mathbb{P}^2$*, a scroll* $\mathbb{F}_n$ *or a scroll blown up in one or two points (including* dP_6*, the del Pezzo surface of degree 6).*

1.4. Thanks. — This note is largely a reworking of original ideas of Iku Nakamura, and MR had access over several years to his work in progress and early drafts of the preprint [**N**]. MR learned the continued fraction tricks here from Jan Stevens (in a quite different context). We are grateful to the organisers of two summer schools at Levico in May 1999 and Lisboa in July 1999 which stimulated our discussion of this material, and to Victor Batyrev for the question that we partially answer in 2.8.4.

1.5. Recent developments. — Since this article first appeared on the e-print server in September 1999 there has been considerable progress in our understanding of the G-Hilbert scheme. The most significant development is the work of Bridgeland, King and Reid [**BKR**] establishing that G-Hilb $\mathbb{C}^3 \to \mathbb{C}^3/G$ is a crepant resolution for a finite (not necessarily Abelian) subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$. In fact [**BKR**] settles many

(1) Homework sheets are on the lecturer's website www.maths.warwick.ac.uk/∼miles.

of the outstanding issues concerning G-Hilb $\mathbb{C}^3$; for instance, an isomorphism between the K theory of G-Hilb $\mathbb{C}^3$ and the representation ring of G is established, and the "dynamic" versus "algebraic" definition of G-Hilb $\mathbb{C}^3$ is settled (see the discussion in Section 4.1 below).

The explicit calculation of the fan Σ of A-Hilb $\mathbb{C}^3$ introduced in the current article enabled AC to establish a geometric construction of the McKay correspondence. Indeed, a certain cookery with the Chern classes of the Gonzalez-Sprinberg and Verdier sheaves $\mathcal{F}_\rho$ (see [**R**] for a discussion) leads to a $\mathbb{Z}$-basis of the cohomology $H^*(Y_\Sigma, \mathbb{Z})$ for which the bijection

$$\Big\{\text{irreducible representations of } A\Big\} \longleftrightarrow \text{ basis of } H^*(Y_\Sigma, \mathbb{Z})$$

holds, with $Y_\Sigma = A$-Hilb $\mathbb{C}^3$ (see [**C1**] for more details). Also, Rebecca Leng's forthcoming Warwick Ph.D. thesis [**L**] extends the explicit calculations in the current article to some non-Abelian subgroups of $\mathrm{SL}(3, \mathbb{C})$.

Our understanding of the construction of G-Hilb $\mathbb{C}^3$ as a variation of GIT quotient of $\mathbb{C}^3/G$ has also improved. Work of King, Ishii and Craw (summarised in [**C2**], Chapter 5) opened the way to a toric treatment of moduli of representations of the McKay quiver (also called moduli of *G-constellations* to stress the link with G-clusters). Initial evidence suggests that these moduli are flops of G-Hilb $\mathbb{C}^3$: every flop of G-Hilb $\mathbb{C}^3$ has been constructed in this way for the quotient of $\mathbb{C}^3$ by the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ (see 1.2) and for the cyclic quotient singularities $\frac{1}{6}(1, 2, 3)$ and $\frac{1}{11}(1, 2, 8)$.

2. Concatenating continued fractions

2.1. Propellor with three blades. — The key to Theorem 1.1 is the observation that easy games with continued fractions provide all the regular triples v_1, v_2, v_3 (see 1.2) among the vectors $f_{i,j}$. First translate the three Newton polygons at e_1, e_2, e_3 to a common vertex, to get the propellor shape of Figure 3, in which three hexants

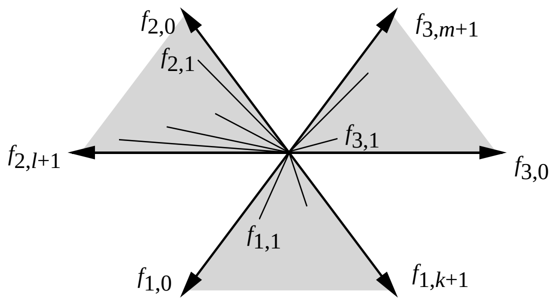

FIGURE 3. "Propellor" with three "blades"

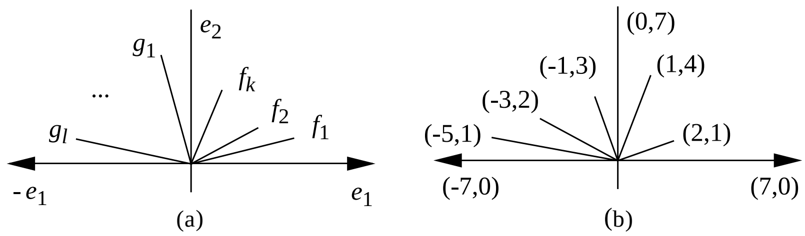

FIGURE 4. Complementary cones $\langle e_1, e_2\rangle$ and $\langle e_2, -e_1\rangle$

(the blades of the propellor) have convex basic subdivisions. The primitive vectors are read in cyclic order

$$f_{1,0}, f_{1,1}, \dots, f_{1,k}, f_{1,k+1} = -f_{2,0}, f_{2,1}, \quad \text{etc.}$$

Inverting any blade (that is, multiplying it by -1) makes the three hexants into a basic subdivision of a half-space. Taking plus or minus all three blades gives a basic subdivision of the plane invariant under -1.

2.2. Two complementary cones. — This digression on well-known material (see for example [**Rie**], §3, pp. 220–3) illustrates several points. Let L be a 2-dimensional lattice, and $e_1, e_2 \in L$ primitive vectors spanning a cone in $L_{\mathbb{R}}$. Then $\mathbb{Z}^2 = \mathbb{Z}\cdot e_1 + \mathbb{Z}\cdot e_2 \subset L$ is a sublattice with cyclic quotient $L/\mathbb{Z}^2 = \mathbb{Z}/r$; assume for the moment that $r > 1$. The *reduced* generator is $f_1 = \frac{1}{r}(\alpha, 1)$ with $1 \leqslant \alpha < r$ and α, r coprime, so that $L = \mathbb{Z}^2 + \mathbb{Z}\cdot\frac{1}{r}(\alpha, 1)$. The continued fraction expansion $r/\alpha = [a_1, \dots, a_k]$ with $a_i \geqslant 2$ gives the convex basic subdivision $\langle e_1, f_1\rangle$, $\langle f_i, f_{i+1}\rangle$, $\langle f_k, e_2\rangle$ in the first quadrant of Figure 4.a.

Repeat the same construction for the cone $\langle e_2, -e_1\rangle$; for this, write the extra generator $\frac{1}{r}(\alpha, 1)$ as $\frac{1}{r}(\alpha e_2, (r-1)(-e_1))$. The reduced normal form is $\frac{1}{r}(1, \beta)$ with $\alpha\beta = (r-1) \bmod r$, or $\beta = 1/(r-\alpha) \bmod r$. The corresponding continued fraction $r/\beta = [b_1, \dots, b_l]$ gives the basic subdivision $e_2, g_1, \dots, g_l, -e_1$ in the top left quadrant of Figure 4.a. (In the literature, this is usually given as $r/(r-\alpha) = [b_l, \dots, b_1]$, but we want this cyclic order.)

Now the vectors $e_1, f_1, \dots, f_k, e_2, g_1, \dots, g_l, -e_1$ form a basic subdivision of the upper half-space of L. The whole trick is the trivial observation that this cannot be convex (downwards) everywhere, so that at e_2,

$$f_k + g_1 = ce_2 \quad \text{with } c \in \mathbb{Z} \text{ and } 0 \leqslant c \leqslant 1. \tag{2.1}$$

For vectors f_k, g_1 in the closed upper half-space, $c = 0$ is only possible if $f_k = e_1$ and $g_1 = -e_1$. Then $r = 1$; this is the "trivial case" with empty continued fractions, at which induction stops. Otherwise, $f_k + g_1 = e_2$. In view of this relation, put a 1 against e_2, and concatenate the two continued fractions as

$$[a_1, a_2, \dots, a_k, 1, b_1, \dots, b_l] \quad (= 0).$$