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RESOLVING 3-DIMENSIONAL TORIC SINGULARITIES

by

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Abstract. — This paper surveys, in the first place, some basic facts from the classification theory of normal complex singularities, including details for the low dimensions 2 and 3. Next, it describes how the toric singularities are located within the class of rational singularities, and recalls their main properties. Finally, it focuses, in particular, on a toric version of Reid's desingularization strategy in dimension three.

1. Introduction

There are certain general *qualitative criteria* available for the rough classification of singularities of complex varieties. The main ones arise:

- from the study of the punctual algebraic behaviour of these varieties (w.r.t. local rings associated to singular points) [*algebraic classification*]
- from an intrinsic characterization for the nature of the possible exceptional loci w.r.t. any desingularization [*rational, elliptic, non-elliptic etc.*]
- from the behaviour of "discrepancies" (for $\mathbb{Q}$-Gorenstein normal complex varieties) [*adjunction-theoretic classification*]

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Algebraic Classification. — At first we recall some fundamental definitions from commutative algebra (cf. [**52**], [**54**]). Let R be a commutative ring with 1. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of R is the supremum of the lengths of all prime ideal chains which are contained in $\mathfrak{p}$, and the *dimension* of R is defined to be

$$\dim(R) := \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \text{ prime ideal of } R\}.$$

R is *Noetherian* if any ideal of it has a finite system of generators. R is a *local ring* if it is endowed with a *unique* maximal ideal $\mathfrak{m}$. A local ring R is *regular* (resp. *normal*) if $\dim(R) = \dim\big(\mathfrak{m}/\mathfrak{m}^2\big)$ (resp. if it is an integral domain and is integrally closed in its field of fractions). A finite sequence $a_1, \ldots, a_\nu$ of elements of a ring R is defined to be a *regular sequence* if a_1 is not a zero-divisor in R and for all i, $i = 2, \ldots, \nu$, a_i is not a zero-divisor of $R/\langle a_1, \ldots, a_{i-1}\rangle$. A Noetherian local ring R (with maximal ideal $\mathfrak{m}$) is called *Cohen-Macaulay* if

$$\mathrm{depth}(R) = \dim(R),$$

where the *depth* of R is defined to be the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. A Cohen-Macaulay local ring R is called *Gorenstein* if

$$\mathrm{Ext}_R^{\dim(R)}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}.$$

A Noetherian local ring R is said to be a *complete intersection* if there exists a regular local ring R', such that $R \cong R'/(f_1, \ldots, f_q)$ for a finite set $\{f_1, \ldots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. The hierarchy by inclusion of the above types of Noetherian local rings is known to be described by the following diagram:

$$\begin{array}{ccc} \{\text{Noetherian local rings}\} & \supset & \{\text{normal local rings}\} \\ \cup & & \cup \\ \{\text{Cohen-Macaulay local rings}\} & & \{\text{regular local rings}\} \\ \cup & & \cap \\ \{\text{Gorenstein local rings}\} & \supset & \{\text{complete intersections (“c.i.'s”)}\} \end{array} \tag{1.1}$$

An arbitrary Noetherian ring R and its associated affine scheme $\mathrm{Spec}(R)$ are called Cohen-Macaulay, Gorenstein, normal or regular, respectively, iff all the localizations $R_{\mathfrak{m}}$ with respect to all the members $\mathfrak{m} \in \text{Max-Spec}(R)$ of the maximal spectrum of R are of this type. In particular, if the $R_{\mathfrak{m}}$'s for all maximal ideals $\mathfrak{m}$ of R are c.i.'s, then one often says that R is a *locally complete intersection* (“l.c.i.”) to distinguish it from the “global” ones. (A *global complete intersection* (“g.c.i.”) is defined to be a ring R of finite type over a field $\boldsymbol{k}$ (i.e., an affine $\boldsymbol{k}$-algebra), such that

$$R \cong \boldsymbol{k}[\mathsf{T}_1.., \mathsf{T}_d] \,/\, (\varphi_1(\mathsf{T}_1, .., \mathsf{T}_d), .., \varphi_q(\mathsf{T}_1, .., \mathsf{T}_d))$$

for q polynomials $\varphi_1, \ldots, \varphi_q$ from $\boldsymbol{k}[\mathsf{T}_1, .., \mathsf{T}_d]$ with $q = d - \dim(R)$). Hence, the above inclusion hierarchy can be generalized for all Noetherian rings, just by omitting in (1.1) the word “local” and by substituting l.c.i.'s for c.i.'s.

We shall henceforth consider only *complex varieties* $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $\boldsymbol{k} = \mathbb{C}$; thus, the punctual algebraic behaviour of X is determined by the stalks $\mathcal{O}_{X,x}$ of its structure sheaf $\mathcal{O}_X$, and X itself is said to have a given *algebraic property* whenever all $\mathcal{O}_{X,x}$'s have the analogous property from (1.1) for all $x \in X$. Furthermore, via the GAGA-correspondence ([**71**], [**30**, § 2]) which preserves the above quoted algebraic properties, we may work within the *analytic category* by using the usual contravariant functor

$$(X, x) \rightsquigarrow \mathcal{O}_{X,x}^{\mathrm{hol}}$$

between the category of isomorphy classes of germs of X and the corresponding category of isomorphy classes of analytic local rings at the marked points x. For a complex variety X and $x \in X$, we denote by $\mathfrak{m}_{X,x}$ the maximal ideal of $\mathcal{O}_{X,x}^{\mathrm{hol}}$ and by

$$\begin{aligned} \text{(1.2)} \qquad \mathrm{Sing}\,(X) &= \left\{ x \in X \;\mid\; \mathcal{O}_{X,x}^{\mathrm{hol}} \text{ is a non-regular local ring} \right\} \\ &= \left\{ x \in X \;\mid\; \dim \left(\mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2 \right) > \dim_x (X) \right\} \end{aligned}$$

its *singular locus.* By a *desingularization* (or *resolution of singularities*) $f : \widehat{X} \to X$ of a non-smooth X, we mean a "full" or "overall" desingularization (if not mentioned), i.e., $\mathrm{Sing}(\widehat{X}) = \varnothing$. When we deal with *partial* desingularizations, we mention it explicitly.

Rational and Elliptic Singularities. — We say that X has (at most) *rational singularities* if there exists a desingularization $f : Y \to X$ of X, such that

$$f_* \mathcal{O}_Y = \mathcal{O}_X$$

(equivalently, Y is normal), and

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leqslant i \leqslant \dim_{\mathbb{C}} X - 1.$$

(The i-th direct image sheaf is defined via

$$U \longmapsto R^i f_* \mathcal{O}_Y (U) := H^i \left(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)} \right)).$$

This definition is independent of the particular choice of the desingularization of X. (Standard example: quotient singularities[(1)] are rational singularities).

We say that a Gorenstein singularity x of X is an *elliptic singularity* if there exists a desingularization $f : Y \to X$ of $x \in X$, such that

$$R^i f_* \mathcal{O}_Y = 0, \quad \forall i, \quad 1 \leqslant i \leqslant \dim_{\mathbb{C}} X - 2,$$

[(1)]The *quotient singularities* are of the form $(\mathbb{C}^r/G, [\mathbf{0}])$, where G is a finite subgroup of $\mathrm{GL}(r, \mathbb{C})$ (without pseudoreflections) acting linearly on $\mathbb{C}^r$, $p : \mathbb{C}^r \to \mathbb{C}^r/G = \mathrm{Spec}(\mathbb{C}[\boldsymbol{z}]^G)$ the quotient map, and $[\mathbf{0}] = p(\mathbf{0})$. Note that

$$\mathrm{Sing}\,(\mathbb{C}^r/G) = p\left(\{\boldsymbol{z} \in \mathbb{C}^r \mid G_{\boldsymbol{z}} \neq \{\mathrm{Id}\}\}\right)$$

(cf. (1.2)), where $G_{\boldsymbol{z}} := \{g \in G \mid g \cdot \boldsymbol{z} = \boldsymbol{z}\}$ is the isotropy group of $\boldsymbol{z} \in \mathbb{C}^r$.

and

$$R^{\dim_{\mathbb{C}} X-1} f_* \mathcal{O}_Y \cong \mathbb{C}.$$

(The definition is again independent of the particular choice of the desingularization).

Adjunction-Theoretic Classification. — If X is a *normal* complex variety, then its Weil divisors can be described by means of "divisorial" sheaves as follows:

Lemma 1.1 **([34,** 1.6]**).** — *For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules the following conditions are equivalent*:
(i) $\mathcal{F}$ *is* reflexive (*i.e.,* $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$, *with* $\mathcal{F}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ *denoting the dual of* $\mathcal{F}$) *and has rank one.*
(ii) *If* X^0 *is a non-singular open subvariety of* X *with* $\operatorname{codim}_X (X \smallsetminus X^0) \geqslant 2$, *then* $\mathcal{F}|_{X^0}$ *is invertible and*

$$\mathcal{F} \cong \iota_* (\mathcal{F}|_{X^0}) \cong \iota_* \iota^* (\mathcal{F}),$$

where $\iota : X^0 \hookrightarrow X$ *denotes the inclusion map.*

The *divisorial sheaves* are exactly those satisfying one of the above conditions. Since a divisorial sheaf is torsion free, there is a non-zero section $\gamma \in H^0(X, Rat_X \otimes_{\mathcal{O}_X} \mathcal{F})$, with

$$H^0(X, Rat_X \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathbb{C}(X) \cdot \gamma,$$

and $\mathcal{F}$ can be considered as a subsheaf of the constant sheaf Rat_X of rational functions of X, i.e., as a special *fractional ideal sheaf.*

Proposition 1.2 **([63,** App. of § 1]**).** — *The correspondence*

$$Cl(X) \ni \{D\} \overset{\delta}{\longmapsto} \{\mathcal{O}_X(D)\} \in \left\{ \begin{array}{c} \textit{divisorial coherent} \\ \textit{subsheaves of } Rat_X \end{array} \right\} \Big/ H^0(X, \mathcal{O}_X^*)$$

with $\mathcal{O}_X(D)$ *defined by sending every non-empty open set* U *of* X *onto*

$$U \longmapsto \mathcal{O}_X(D)(U) := \left\{ \varphi \in \mathbb{C}(X)^* \mid (\operatorname{div}(\varphi) + D)|_U \geqslant 0 \right\} \cup \{0\},$$

is a bijection, and induces a $\mathbb{Z}$*-module isomorphism. In fact, to avoid torsion, one defines this* $\mathbb{Z}$*-module structure by setting*

$$\delta(D_1 + D_2) := (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee} \text{and } \delta(\kappa D) := \mathcal{O}_X(D)^{[\kappa]} = \mathcal{O}_X(\kappa D)^{\vee\vee},$$

for any Weil divisors D, D_1, D_2 *and* $\kappa \in \mathbb{Z}$.

Let now $\Omega_{\operatorname{Reg}(X)/\mathbb{C}}$ be the sheaf of regular 1-forms, or Kähler differentials, on

$$\operatorname{Reg}(X) = X \smallsetminus \operatorname{Sing}(X) \overset{\iota}{\hookrightarrow} X,$$

(cf. [**36**, § 5.3]) and for $i \geqslant 1$, let us set

$$\Omega^i_{\operatorname{Reg}(X)/\mathbb{C}} := \bigwedge^i \Omega_{\operatorname{Reg}(X)/\mathbb{C}}.$$

The unique (up to rational equivalence) Weil divisor K_X, which maps under δ to the *canonical divisorial sheaf*

$$\omega_X := \iota_* \left(\Omega^{\dim_{\mathbb{C}}(X)}_{\mathrm{Reg}(X)/\mathbb{C}} \right),$$

is called the *canonical divisor* of X. Another equivalent interpretation of ω_X, when X is Cohen-Macaulay, can be given by means of the Duality Theory (see [**32**], [**29**]). If $\mathbb{D}^+_c(\mathcal{O}_X)$ denotes the derived category of below bounded complexes whose cohomology sheaves are coherent, then there exists a *dualizing complex*(2) $\omega^\bullet_X \in \mathbb{D}^+_c(\mathcal{O}_X)$ over X. If X is Cohen-Macaulay, then the i-th cohomology sheaf $\mathcal{H}^i(\omega^\bullet_X)$ vanishes for all $i \in \mathbb{Z} \smallsetminus \{-\dim_{\mathbb{C}}(X)\}$, and $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega^\bullet_X)$. This leads to the following:

Proposition 1.3. — *A normal complex variety X is Gorenstein if and only if it is Cohen-Macaulay and ω_X is invertible.*

Proof. — If X is Gorenstein, then $\mathcal{O}_{X,x}$ satisfies the equivalent conditions of [**54**, Thm. 18.1], for all $x \in X$. This means that $\mathcal{O}_{X,x}$ (as Noetherian local ring) is a dualizing complex for itself (cf. [**32**, Ch. V, Thm. 9.1, p. 293]). Since dualizing complexes are unique up to tensoring with an invertible sheaf, say L, over X, shifted by an integer n (cf. [**32**, Ch. V, Cor. 2.3, p. 259]), we shall have $\omega^\bullet_X \cong \mathcal{O}^\bullet_{X,x} \otimes L[n]$. Hence, ω_X itself will be also invertible. The converse follows from the isomorphisms $\omega_X \cong \mathcal{H}^{-\dim_{\mathbb{C}}(X)}(\omega^\bullet_X)$ and $\omega_{X,x} \cong \mathcal{O}_{X,x}$, for all $x \in X$. (Alternatively, one may use the fact that $x \in X$ is Gorenstein iff $\mathcal{O}_{X,x}$ is Cohen-Macaulay and $H^{\dim_{\mathbb{C}}(X)}_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})$ is a dualizing module for it, cf. [**29**, Prop. 4.14, p. 65]. The classical duality [**29**, Thm. 6.3, p. 85], [**32**, Ch. V, Cor. 6.5, p. 280], combined with the above uniqueness argument, gives again the required equivalence). □

Theorem 1.4 **(Kempf [43, p. 50], Elkik [23], [24], Bingener-Storch [5])**
Let X a normal complex variety of dimension $\geqslant 2$. Then

$$\begin{pmatrix} X \text{ has at most} \\ \text{rational singularities} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} X \text{ is Cohen-Macaulay} \\ \text{and } \omega_X \cong f_* \omega_Y \end{pmatrix},$$

where $f : Y \longrightarrow X$ is any desingularization of X.

(Note that, if $E = f^{-1}(\mathrm{Sing}(X))$ and $\iota : \mathrm{Reg}(X) \hookrightarrow X$, $j : Y \smallsetminus E \hookrightarrow Y$ are the natural inclusions, then by the commutative diagram

$$\begin{array}{ccc} Y \smallsetminus E & \overset{j}{\hookrightarrow} & Y \\ \downarrow \wr & & \downarrow f \\ \mathrm{Reg}(X) & \overset{\iota}{\hookrightarrow} & X \end{array}$$

(2) There is a canonical morphism $\omega_X[\dim_{\mathbb{C}}(X)] \to \omega^\bullet_X$ which is a quasi-isomorphism iff X is Cohen-Macaulay.