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PRODUCING GOOD QUOTIENTS BY EMBEDDING INTO TORIC VARIETIES

by

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Abstract. — Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial algebraic variety X. Suppose that X has the A_2-property, that means any two points of X admit a common affine open neighbourhood in X. We prove the following embedding theorem: Let $U_1, \ldots, U_r \subset X$ be T-invariant open subsets with good quotients $U_i \to U_i /\!\!/ T$ such that the $U_i /\!\!/ T$ are A_2-varieties. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that each U_i is of the form $U_i = W_i \cap X$ with a toric open subset $W_i \subset Z$ admitting a good quotient $W_i \to W_i /\!\!/ T$. This result applies in particular to the family of open subsets $U \subset X$ that are maximal with respect to saturated inclusion among all open subsets admitting a good A_2-quotient space. In the appendix to this article we survey some general results on embeddings into toric varieties and prevarieties.

Introduction

This article deals with toric varieties as ambient spaces in algebraic geometry. We consider actions of algebraic tori T on a $\mathbb{Q}$-factorial (e.g. smooth) algebraic variety X and show that the problem of constructing good quotients for such an action extends to a purely toric problem of a suitable ambient toric variety of X, provided of course that X and the quotient varieties in question are embeddable into toric varieties.

Let us recall the basic notions and some background. A good quotient for the action of an algebraic torus T on a variety X is a T-invariant affine regular map $p\colon X \to X /\!\!/ T$ such that the natural homomorphism $\mathcal{O}_{X/\!\!/T} \to p_*(\mathcal{O}_X)^T$ is an isomorphism. In general, the whole X need not admit a good quotient, but there always exist nonempty open T-invariant subsets $U \subset X$ with a good quotient $U \to U /\!\!/ T$. It is one of the central problems in Geometric Invariant Theory to describe or even to construct all these open subsets.

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In the special case of toric varieties the above problem can be solved: Let Z be a toric variety, and let T be a subtorus of the big torus $T_Z \subset Z$. The description of Z in terms of its fan allows to figure out explicitly all the toric open $W \subset Z$ admitting a good quotient $W \to W/\!\!/T$, see [**9**], [**18**] and also Section 1. Moreover, every further open subset of Z admitting a good quotient by the action of T occurs as a saturated subset of one of these W. A different but also combinatorial approach for $Z = \mathbb{P}_n$ is presented in [**3**].

We shall show that in principle the general problem of constructing good quotients for torus actions can be reduced to the toric setting by means of embedding. Of course, in this approach one has to restrict oneself to embeddable spaces. In view of Włodarczyk's theorem [**20**], this amounts to considering spaces Y with the A_2-property: Any two points of Y admit a common affine open neighbourhood in Y. Our main result is the following, see Theorem 2.4:

Theorem. — *Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial A_2-variety X, and suppose that the T-invariant open subsets $U_1, \ldots, U_r \subset X$ admit good quotients $U_i \to U_i/\!\!/T$ with A_2-varieties $U_i/\!\!/T$. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that each U_i is of the form $U_i = W_i \cap X$ with a toric open subset $W_i \subset Z$ admitting a good quotient $W_i \to W_i/\!\!/T$.*

This applies to the general problem: It suffices to consider the $(T, 2)$-maximal subsets of a given T-variety X, i.e., the invariant open subsets $U \subset X$ that admit a good quotient with an A_2-variety $U/\!\!/T$ and do not occur as a saturated subset of some properly larger U' having the same properties. Święcicka showed that the family of all $(T, 2)$-maximal subsets of X is finite [**19**]. Consequently, we obtain, see Corollary 2.6:

Corollary. — *Let an algebraic torus T act effectively on a $\mathbb{Q}$-factorial A_2-variety X. Then there exists a T-equivariant closed embedding $X \hookrightarrow Z$ into a smooth toric variety Z on which T acts as a subtorus of the big torus such that every $(T, 2)$-maximal open $U \subset X$ is of the form $U = W \cap X$ with a toric open subset $W \subset Z$ admitting a good quotient $W \to W/\!\!/T$.*

Note that this generalizes the following result due to Święcicka [**19**]: If the torus T acts on a smooth projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$ and $U \subset X$ is $(T, 2)$-maximal, then there is a T-equivariant embedding $X \subset \mathbb{P}_n$ such that $U = W \cap X$ with a $(T, 2)$-maximal and hence $\mathbb{T}_n$-invariant $W \subset \mathbb{P}_n$.

The present article is organized as follows: In Section 1 we introduce the basic notions and discuss some known results on good quotients for toric varieties. Section 2 is devoted to giving the precise formulation of our main result. In Section 3 we provide the techniques for the proof of our main result which is performed in Section 4. Finally, in the appendix, we survey some general results on embeddings into toric varieties and prevarieties.

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1. Good quotients of toric varieties

In this section we discuss some well-known results on good quotients for subtorus actions on toric varieties. As we shall need this later, we perform our fixing of terminology in the more general setting of possibly non separated prevarieties.

Throughout the whole article we work over an algebraically closed field $\mathbb{K}$. A *toric prevariety* is a normal (algebraic) prevariety X (over $\mathbb{K}$) together with an algebraic torus $T_X \subset X$ such that T_X is open in X and a regular action $T_X \times X \to X$ that extends the group structure of $T_X \subset X$. We refer to $T_X \subset X$ as the big torus of X. A *toric variety* is a separated toric prevariety.

A *toric morphism* of two toric prevarieties X, X' is a regular map $f\colon X \to X'$ that restricts to a group homomorphism $\varphi\colon T_X \to T_{X'}$ of the respective big tori satisfying $f(t\cdot x) = \varphi(t)\cdot x$ for all $(t,x) \in T \times X$. Similarly to the separated case, the category of toric prevarieties can be described by certain combinatorial data, see [**1**].

A *good prequotient* for a regular action $G \times X \to X$ of a reductive group on a prevariety X is a G-invariant affine regular map $p\colon X \to X/\!\!/G$ of prevarieties such that the canonical map $\mathcal{O}_{X/\!\!/G} \to p_*(\mathcal{O}_X)^G$ is an isomorphism. A good prequotient $p\colon X \to X/\!\!/G$ is called *geometric*, if it separates orbits. If both spaces X and $X/\!\!/G$ are separated, then we speak of a *good* or a *geometric quotient*.

Now, let X be a toric prevariety. As announced above, we consider actions of subtori T of the big torus $T_X \subset X$. Concerning good prequotients of such subtorus actions, the first observation is, see e.g. [**1**, Corollary 6.5]:

Remark 1.1. — If the action of $T \subset T_X$ has a good prequotient $p\colon X \to X/\!\!/T$, then the quotient space $X/\!\!/T$ inherits the structure of a toric prevariety such that p becomes a toric morphism.

In our article the following property of varieties will play a central rôle: We say that a variety X has the *A_2-property*, if any two points $x, x' \in X$ admit a common affine open neighbourhood in X. This notion is due to J. Włodarczyk. In [**20**] he proves among other things that a normal variety X admits a closed embedding into a toric variety if and only if X is A_2.

The next statement is a simple, but useful toric version of [**4**, Theorem C]. It shows that the A_2-property is in a natural way connected with good quotients of toric varieties:

Proposition 1.2. — *Let X be a toric variety with big torus $T_X \subset X$. For every subtorus $T \subset T_X$ the following statements are equivalent:*

i) *The action of T on X has a good quotient $X \to X /\!\!/ T$.*

ii) *Any two points $x, x' \in X$ with closed T_X-orbit have a common T-invariant affine neighbourhood in X.*

Proof. — If the action of T on X admits a good quotient $X \to X /\!\!/ T$, then the quotient space inherits the structure of a toric variety and hence has the A_2-property. Since the quotient map $X \to X /\!\!/ T$ is affine and T-invariant, it follows that X fullfills Condition ii).

Now suppose that ii) holds. According to [**4**, Theorem C], we only have to show that any two points of X have a common affine T-invariant neighbourhood in X. So, given $z, z' \in X$, choose

$$x \in \overline{T_X \cdot z}, \qquad x' \in \overline{T_X \cdot z'}$$

such that the orbits $T_X \cdot x$ and $T_X \cdot x'$ are closed in X. By assumption, there exists a T-invariant affine open $U \subset X$ with $x, x' \in U$. Consider the sets

$$S := \{t \in T_X;\ t \cdot z \in U\}, \qquad S' := \{t \in T_X;\ t \cdot z' \in U\}.$$

These are non empty open subsets of T_X and hence we have $S \cap S' \neq \varnothing$. Let $t \in S \cap S'$. Then $t^{-1} \cdot U$ is the desired common affine neighbourhood of the points z and z'. □

Finally, we characterize existence of good quotients in terms of fans. For the terminology, see [**8**]. Let Δ be a fan in some lattice N, and let $L \subset N$ be a primitive sublattice. Then Δ defines a toric variety X, and L corresponds to a subtorus T of the big torus $T_X \subset X$.

Up to elementary convex geometry, the following statement is a reformulation of a well-known characterization obtained by J. Święcicka [**18**, Theorem 4.1] and, independently, by H. Hamm [**9**, Theorem 4.7]. For convenience, we present here a direct proof in our setting.

Proposition 1.3. — *The action of T on X admits a good quotient if and only if any two different maximal cones of Δ can be separated by an L-invariant linear form on N.*

Proof. — First suppose that the action of T has a good quotient $q\colon X \to X'$. Then X' inherits the structure of a toric variety such that q becomes a toric morphism. So we may assume that q arises from a map of fans $Q\colon N \to N'$ from Δ to a fan Δ' in a lattice N'.

Note that the sublattice $L \subset N$ is contained in $\ker(Q)$. Let $Q_{\mathbb{R}}\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be the linear map of real vector spaces associated to $Q\colon N \to N'$. We claim that there are bijections of the sets $\Delta^{\max}$ and $(\Delta')^{\max}$ of maximal cones:

$$\Delta^{\max} \to (\Delta')^{\max}, \qquad \sigma \mapsto Q_{\mathbb{R}}(\sigma), \tag{1}$$

$$(\Delta')^{\max} \to \Delta^{\max}, \qquad \sigma' \mapsto Q_{\mathbb{R}}^{-1}(\sigma') \cap |\Delta| \tag{2}$$

To check that the first map is well-defined, let $\sigma \in \Delta^{\max}$. Then the image $Q_{\mathbb{R}}(\sigma)$ is contained in some maximal cone $\sigma' \in \Delta'$. In particular, $q(X_\sigma) \subset X_{\sigma'}$ holds. Since

q is affine, the inverse image $q^{-1}(X_{\sigma'})$ is an affine invariant chart of X, and hence necessarily equals X_σ. Since q is in addition surjective, we must have $q(X_\sigma) = X_{\sigma'}$. This means $Q_{\mathbb{R}}(\sigma) = \sigma'$. So we see that (1) is well defined.

To see that also the second map is well defined, let $\sigma' \in (\Delta')^{\max}$. The inverse image of the associated affine chart $X_{\sigma'} \subset X'$ is given by the general formula

$$q^{-1}(X_{\sigma'}) = \bigcup_{\tau \in \Delta; Q_{\mathbb{R}}(\tau) \subset \sigma'} X_\tau.$$

Since q is affine, this inverse image is an affine invariant chart X_σ given by some cone $\sigma \in \Delta$. It follows that

$$\sigma = \operatorname{cone}(\tau \in \Delta;\ Q_{\mathbb{R}}(\tau) \subset \sigma') = Q_{\mathbb{R}}^{-1}(\sigma') \cap |\Delta|.$$

We still have to check that σ is maximal. By surjectivity of q, we see $Q_{\mathbb{R}}(\sigma) = \sigma'$ holds. Now assume, that $\sigma \subset \tau$ for some $\tau \in \Delta^{\max}$. As seen above, $Q_{\mathbb{R}}(\tau)$ is a maximal cone of Δ'. Since $Q_{\mathbb{R}}(\tau)$ contains the maximal cone σ', we get $Q_{\mathbb{R}}(\tau) = \sigma'$. By definition of σ, this implies $\tau = \sigma$. So, also (2) is well defined.

Obviously, the maps (1) and (2) are inverse to each other. We use them to find separating linear forms. Let σ_1, σ_2 be two different maximal cones. Then the maximal cones $\sigma_i' := Q_{\mathbb{R}}(\sigma_i)$ of Δ' can be separated by a linear form u' on N', i.e.,

$$u'|_{\sigma_1'} \geq 0, \qquad u'|_{\sigma_2'} \leq 0, \qquad (u')^\perp \cap \sigma_1' = (u')^\perp \cap \sigma_2' = \sigma_1' \cap \sigma_2'.$$

Now consider the linear form $u := u' \circ Q$. Then u is L-invariant, nonnegative on σ_1 and nonpositive on σ_2. Using (1) and (2) we obtain:

$$\begin{aligned} u^\perp \cap \sigma_i &= Q_{\mathbb{R}}^{-1}((u')^\perp) \cap (Q_{\mathbb{R}}^{-1}(\sigma_i') \cap |\Delta|) \\ &= Q_{\mathbb{R}}^{-1}((u')^\perp \cap \sigma_i') \cap |\Delta| \\ &= Q_{\mathbb{R}}^{-1}(\sigma_1' \cap \sigma_2') \cap |\Delta| \\ &= (Q_{\mathbb{R}}^{-1}(\sigma_1') \cap |\Delta|) \cap (Q_{\mathbb{R}}^{-1}(\sigma_2') \cap |\Delta|) \\ &= \sigma_1 \cap \sigma_2. \end{aligned}$$

Now suppose that any two different maximal cones of Δ can be separated by an L-invariant linear form on N. Let $P\colon N \to N/L$ denote the projection. We claim that the projected cones $P_{\mathbb{R}}(\sigma)$, where σ runs through the maximal cones of Δ, are the maximal cones of a quasifan Σ in N/L, i.e., this Σ behaves almost like a fan, merely its cones need not be strictly convex.

To verify this claim, we have to find for any two $\sigma_1' := P_{\mathbb{R}}(\sigma_1)$ and $\sigma_2' := P_{\mathbb{R}}(\sigma_2)$, where $\sigma_1, \sigma_2 \in \Delta^{\max}$, a separating linear form. By assumption, there is an L-invariant linear form u on N that separates σ_1 and σ_2. Let u' denote the linear form on N/L with $u = u' \circ P$. Then u' is nonnegative on σ_1' and nonpositive on σ_2'. Moreover, we