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SPECIAL MCKAY CORRESPONDENCE

by

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Dedicated to Professor Riemenschneider for his 60th birthday

Abstract. — There are many generalizations of the McKay correspondence for higher dimensional Gorenstein quotient singularities and there are some applications to compute the topological invariants today. But some of the invariants are completely different from the classical invariants, in particular for non-Gorenstein cases. In this paper, we would like to discuss the McKay correspondence for 2-dimensional quotient singularities via "special" representations which gives the classical topological invariants and give a new characterization of the special representations for cyclic quotient singularities in terms of combinatorics.

Contents

1. McKay correspondence

The McKay correspondence is originally a correspondence between the topology of the minimal resolution of a 2-dimensional rational double point, which is a quotient singularity by a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, and the representation theory (irreducible representations or conjugacy classes) of the group G. We can see the

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correspondence via Dynkin diagrams, which came from McKay's observation in 1979 ([**16**]).

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, then the quotient space $X := \mathbb{C}^2/G$ has a rational double point at the origin. As there exists the minimal resolution $\widetilde{X}$ of the singularity, we have the exceptional divisors E_i. The dual graph of the configuration of the exceptional divisors is just the Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

On the other hand, we have the set of the irreducible representations ρ_i of the group G up to isomorphism and let ρ be the natural representation in $\mathrm{SL}(2,\mathbb{C})$. The tensor product of these representations

$$\rho_i \otimes \rho = \bigoplus_{j=0}^{r} a_{ij}\rho_j,$$

where ρ_0 is the trivial representation and r is the number of the non-trivial irreducible representations, gives a set of integers a_{ij} and it determines the Cartan matrix which defines the Dynkin diagram. (1)

Then we have a one-to-one numerical correspondence between non-trivial irreducible representations $\{\rho_i\}$ and irreducible exceptional curves $\{E_i\}$, that is, the intersection matrix of the exceptional divisors is the opposite of the Cartan matrix.

This phenomenon was explained geometrically in terms of vector bundles on the minimal resolution by Gonzalez-Sprinberg and Verdier ([**8**]) (2) by case-by-case computations in 1983. In 1985, Artin and Verdier ([**1**]) proved this more generally with reflexive modules and this theory was developed by Esnault and Knörrer ([**5**], [**6**]) for more general quotient surface singularities. After Wunram ([**21**]) constructed a nice generalized McKay correspondence for any quotient surface singularities in 1986 in his dissertation, Riemenschneider introduced the notion of "special representation etc." and made propaganda for the more generalized McKay correspondence (cf. [**18**]). (3)

In dimension three, we have several "McKay correspondences" but they are just bijections between two sets: Let X be the quotient singularity $\mathbb{C}^3/G$ where G is a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$. Then X has a Gorenstein canonical singularity of index 1 but not a terminal singularity. It is known that there exist crepant resolutions $\widetilde{X}$ of this singularity. The crepant resolution is a minimal resolution and preserves the triviality of the canonical bundle in this case.

As for the McKay correspondence, the followings are known:

(1) (Ito-Reid [**12**]) There exists a base of cohomology group $H^{2i}(\widetilde{X},\mathbb{Q})$, indexed by the conjugacy classes of "age" i in G.

(1) More precisely, the Cartan matrix is defined as the matrix $2E - A$, where E is the $r \times r$ identity matrix and $A = \{a_{ij}\}$ $(i, j \neq 0)$.

(2) They gave the name *McKay correspondence* (in French, *la correspondance de McKay*) in this paper!

(3) Similar generalization for $G \subset \mathrm{GL}(2,\mathbb{C})$ was obtained by Gonzalez-Sprinberg and the related topics were discussed in [**7**].

(2) (Ito-Nakajima [**10**]) There exists a base of Grothendieck group $K(\widetilde{X})$, indexed by the irreducible representations of G, when G is a finite abelian subgroup.

(3) (Bridgeland-King-Reid [**3**]) There exists an equivalence between the derived category $D(\widetilde{X})$ and the equivariant dereived category $D^G(\mathbb{C}^3)$ for any finite subgroup.

Remark 1.1. — In (1), the **age** of $g \in G$ is defined as follows: After diagonalization, if $g^r = 1$, we obtain $g' = \mathrm{diag}(\varepsilon^a, \varepsilon^b, \varepsilon^c)$ where ε is a primitive r-th root of unity. Then $\mathrm{age}(g) := (a+b+c)/r$. For the identity element *id*, we define age(*id*)= 0 and all ages are integers if $G \subset \mathrm{SL}(3, \mathbb{C})$.

The correspondence (2) can be included in (3), but note that the 2-dimensional numerical McKay correspondence can be explained very clearly as a corollary of the result in [**10**].

As a generalization of the first McKay correspondence (1), we have a precise correspondence for each $2i$-th cohomology with conjugacy classes of age i for any $i = 1, \ldots, n-1$ in dimension n which was given by Batyrev and Kontsevich via "motivic integral" under the assumption of the existence of a crepant resolution, and this idea was developed to "string theoretic cohomology" for all quotient singularities (cf. [**2**]).

And we can see that the string theoretic Euler number of the resolution is the same as the order of the acting group G in case $G \subset \mathrm{GL}(n, \mathbb{C})$, but it is different from the usual topological Euler number of the minimal resolution. Of course, it is very interesting to consider the geometrical meaning of these new invariants.

By the way, in (2) we don't have such a difference among representations as *age*. But the author is interested in the relation between the group theory and the classical topological invariants. Then we would like to remind the reader of the notion of special representations which gives some differences between irreducible representations. The special representations were defined by Riemenschneider and Wunram ([**18**]); each of the special irreducible respresentations corresponds to an exceptional divisor of the minimal resolution of a 2-dimensional quotient singularity.

In particular, we would like to discuss special representations and the minimal resolution for quotient surface singularities from now on. Around 1996, Nakamura and the author showed another way to the McKay correspondence with the help of the G-Hilbert scheme, which is a 2-dimensional G-fixed set of the usual Hilbert scheme of $|G|$-points on $\mathbb{C}^2$ and isomorphic to the minimal resolution. Kidoh ([**14**]) proved that the G-Hilbert scheme for general cyclic surface singularities is the minimal resolution. Then Riemenschneider checked the cyclic case and conjectured that the representations which are given by the Ito-Nakamura type McKay correspondence via G-Hilbert scheme are just special representations in 1999 ([**19**]) and this conjecture was proved by A. Ishii recently ([**9**]). In this paper, we will give another characterization of the special representations by combinatorics for the cyclic quotient case, using results on the G-Hilbert schemes.

As a colorful introduction to the McKay correspondence, the author would like to recommend a paper presented at the Bourbaki seminar by Reid ([**17**]) and also on the Web page (http://www.maths.warwick.ac.uk/ ∼miles/McKay), one can find some recent papers related to the McKay correspondence.

This paper is organized as follows: In this section, we already gave a brief history of the McKay correspondence and we will discuss the special representations and the generalized McKay correspondence in the following section. In section three, we treat G-Hilbert schemes as a resolution of singularities, consider the relation with the toric resolution in the cyclic case, and show how to find the special representations by combinatorics. In the final section, we will discuss an example and related topics.

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2. Special representations

In this section, we will discuss the special representations. Let G be a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$, that is, the action of the group G is free outside the origin, and ρ be a representation of G on V. G acts on $\mathbb{C}^2 \times V$ and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/G$ which can be extended to a reflexive sheaf $\mathcal{F}$ on $X\colon = \mathbb{C}^2/G$.

For any reflexive sheaf $\mathcal{F}$ on a rational surface singularity X and the minimal resolution $\pi\colon \widetilde{X} \to X$, we define a sheaf $\widetilde{\mathcal{F}}\colon = \pi^*\mathcal{F}/\text{torsion}$.

Definition 2.1 **([5])**. — The sheaf $\widetilde{\mathcal{F}}$ is called a *full sheaf* on $\widetilde{X}$.

Theorem 2.2 **([5])**. — *A sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ is a full sheaf if the following conditions are fulfilled:*

(1) *$\widetilde{\mathcal{F}}$ is locally free,*
(2) *$\widetilde{\mathcal{F}}$ is generated by global sections,*
(3) *$H^1(\widetilde{X}, \widetilde{\mathcal{F}}^\vee \otimes \omega_{\widetilde{X}}) = 0$, where $\vee$ means the dual.*

Note that a sheaf $\widetilde{\mathcal{F}}$ is indecomposable if and only if the corresponding representation ρ is irreducible. Therefore we obtain an indecomposable full sheaf $\widetilde{\mathcal{F}}_i$ on $\widetilde{X}$ for each irreducible representation ρ_i, but in general, the number of the irreducible representations is larger than that of irreducible exceptional components. Therefore Wunram and Riemenschneider introduced the notion of speciality for full sheaves:

***Definition 2.3* ([18]).** — A full sheaf is called *special* if and only if

$$H^1(\widetilde{X}, \widetilde{\mathcal{F}}^\vee) = 0.$$

A reflexive sheaf $\mathcal{F}$ on X is *special* if $\widetilde{\mathcal{F}}$ is so.

A representation ρ is *special* if the associated reflexive sheaf $\mathcal{F}$ on X is special.

With these definitions, the following equivalent conditions for the speciality hold:

***Theorem 2.4* ([18], [21])**

(1) $\widetilde{\mathcal{F}}$ *is special* $\Longleftrightarrow$ $\widetilde{\mathcal{F}} \otimes \omega_{\widetilde{X}} \to [(\mathcal{F} \otimes \omega_{\widetilde{X}})^{\vee\vee}]^\sim$ *is an isomorphism,*

(2) $\mathcal{F}$ *is special* $\Longleftrightarrow$ $\mathcal{F} \otimes \omega_{\widetilde{X}}/$*torsion is reflexive,*

(3) ρ *is a special representation* $\Longleftrightarrow$ *the map* $(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V)^G$ *is surjective.*

Then we have the following nice generalized McKay correspondence for quotient surface singularities:

***Theorem 2.5* ([21]).** — *There is a bijection between the set of special non-trivial indecomposable reflexive modules* $\mathcal{F}_i$ *and the set of irreducible components* E_i *via* $c_1(\widetilde{\mathcal{F}}_i)E_j = \delta_{ij}$ *where* c_1 *is the first Chern class, and also a one-to-one correspondence with the set of special non-trivial irreducible representations.*

As a corollary of this theorem, we get back the original McKay correspondence for finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ because in this case all irreducible representations are special.

3. G-Hilbert schemes and combinatorics

In this section, we will discuss G-Hilbert schemes and a new way to find the special representations for cyclic quotient singularities by combinatorics.

The Hilbert scheme of n points on $\mathbb{C}^2$ can be described as a set of ideals:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \mid I \text{ ideal}, \dim \mathbb{C}[x,y]/I = n\}.$$

It is a $2n$-dimensional smooth quasi–projective variety. The G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ was introduced in the paper by Nakamura and the author ([**11**]) as follows:

$$\mathrm{Hilb}^G(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \mid I\ G\text{-invariant ideal}, \mathbb{C}[x,y]/I \cong \mathbb{C}[G]\},$$

where $|G| = n$. This is a union of components of fixed points of G-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and in fact it is just the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. It was proved for $G \subset \mathrm{SL}(2,\mathbb{C})$ in [**11**] first by the properties of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and finite group action of G and a McKay correspondence in terms of ideals of G-Hilbert schemes was stated.