Séminaires & Congrès
6, 2002, p. 227–247

LECTURES ON HEIGHT ZETA FUNCTIONS OF TORIC VARIETIES

by

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Abstract. — We explain the main ideas and techniques involved in recent proofs of asymptotics of rational points of bounded height on toric varieties.

1. Introduction

Toric varieties are an ideal testing ground for conjectures: their theory is sufficiently rich to reflect general phenomena and sufficiently rigid to allow explicit combinatorial computations. In these notes I explain a conjecture in arithmetic geometry and describe its proof for toric varieties.

Acknowledgments. — I would like to thank the organizers of the Summer School for the invitation. The results concerning toric varieties were obtained in collaboration with V. Batyrev. It has been a great pleasure and privilege to work with A. Chambert-Loir, B. Hassett and M. Strauch — I am very much indebted to them. My research was partially supported by the NSA.

1.1. Counting problems

Example 1.1.1. — Let $X \subset \mathbb{P}^n$ be a smooth hypersurface given as the zero set of a homogeneous form f of degree d (with coefficients in $\mathbb{Z}$). Let

$$N(X,B) = \#\{\boldsymbol{x} \mid f(\boldsymbol{x}) = 0, \max(|x_j|) \leqslant B\}$$

(where $\boldsymbol{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}/(\pm 1)$ with $\gcd(x_j) = 1$) be the number of $\mathbb{Q}$-rational points on X of "height" $\leqslant B$. Heuristically, the probability that f represents 0 is about B^{-d} and the number of "events" about B^{n+1}. Thus we expect that

$$\lim_{B\to\infty} N(X,B) \sim B^{n+1-d}.$$

2000 ***Mathematics Subject Classification.*** — 14G05, 11D45, 14M25, 11D57.
Key words and phrases. — Rational points, heights, toric varieties, zeta functions.

This can be proved by the circle method, at least when $n \gg 2^d$. The above heuristic leads to a natural trichotomy, corresponding to the possibilities when $n+1-d$ positive, zero or negative. In the first case we expect many rational points on X, in the third case very few and in the intermediate case we don't form an opinion.

Example 1.1.2. — Let $X \subset \mathbb{P}^n \times \mathbb{P}^n$ be a hypersurface given as the zero set of a bihomogeneous diagonal form of bidegree (d_1, d_2):

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{n} a_k x_k^{d_1} \cdot y_k^{d_2},$$

with $a_k \in \mathbb{Z}$. Each pair of positive integers $L = (l_1, l_2)$ defines a counting function on rational points $X(\mathbb{Q})$ by

$$N(X, L, B) = \#\{(\boldsymbol{x}, \boldsymbol{y}) \mid f(\boldsymbol{x}, \boldsymbol{y}) = 0,\, \max(|x_i|)^{l_1} \cdot \max(|y_j|)^{l_2} \leqslant B\}$$

(where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^{(n+1)}/(\pm 1)$ with $\gcd(x_i) = \gcd(y_j) = 1$). Heuristics as above predict that the asymptotic should depend on the vector

$$-K = (n+1-d_1, n+1-d_2)$$

and on the location of L with respect to $-K$.

An interesting open problem is, for example, the case when $(d_1, d_2) = (1, 2), n = 3$ and $L = (3, 2)$. Notice that this variety is a compactification of the affine space. For appropriate a_k one expects $\sim B \log(B)$ rational points of height bounded by B.

Trying to systematize such examples one is naturally lead to the following problems:

Problem 1.1.3. — Let $X \subset \mathbb{P}^n$ be an algebraic variety over a number field. Relate the asymptotics of rational points of bounded height to geometric invariants of X.

Problem 1.1.4. — Develop analytic techniques proving such asymptotics.

1.2. Zariski density. — Obviously, not every variety is a hypersurface in a projective space or product of projective spaces. To get some systematic understanding of the distribution of rational points we need to use ideas from classification theories of algebraic varieties. On a very basic level (smooth projective) algebraic varieties are distinguished according to the ampleness of the canonical class: Fano varieties (big anticanonical class), varieties of general type (big canonical class) and varieties of intermediate type (neither). The conjectures of Bombieri-Lang-Vojta predict that on varieties of general type the set of rational points is not Zariski dense (see [**46**]). Faltings proved this for subvarieties of abelian varieties ([**16**]). It is natural to ask for a converse. As the examples of Colliot-Thélène, Swinnerton-Dyer and Skorobogatov suggest (see [**11**]), the most optimistic possibility would be: if X does not have finite étale covers which dominate a variety of general type then there exists a finite extension E/F such that $X(E)$ is Zariski dense in X. In particular, this should hold for Fano varieties. I have no idea how to prove this for a general smooth quintic hypersurface in $\mathbb{P}^5$. Quartic hypersurfaces in $\mathbb{P}^4$ are treated in [**22**] (see also [**23**]).

Clearly, we need Zariski density of rational points on X before attempting to establish a connection between the global geometry of X and $X(F)$. Therefore, we will focus on varieties birational to the projective space or possessing a large group of automorphisms so that rational points are a priori dense, at least after a finite extension. In addition to allowing finite field extensions we will need to restrict to some appropriate Zariski open subsets.

Example 1.2.1. — Let X be the cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ over $\mathbb{Q}$. We expect $\sim B(\log(B))^3$ rational points of height $\max(|x_j|) \leqslant B$. However, on the lines like $x_0 = -x_1$ and $x_2 = -x_3$ we already have $\sim B^2$ rational points. Numerical experiments in [**39**] confirm the expected growth rate on the complement to the lines; and Heath-Brown proved the upper bound $O(B^{4/3+\epsilon})$ [**24**]. Thus the asymptotic of points on the whole X will be dominated by the contribution from lines, and it is futile to try to read off geometric invariants of X from what is happening on the lines.

Such Zariski closed subvarieties will be called *accumulating*. Notice that this notion may depend on the projective embedding.

1.3. Results. — Let X be a smooth projective algebraic variety over a number field F and L a very ample line bundle on X. It defines an embedding $X \hookrightarrow \mathbb{P}^n$. Fix a "height" on the ambient projective space. For example, we may take

$$H(\boldsymbol{x}) := \prod_v \max_j(|x_j|_v),$$

where $\boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{P}^n(F)$ and the product is over all (normalized) valuations of F. To highlight the choice of the height we will write $\mathcal{L}$ for the pair (L-embedding, height). We get an induced (exponential) *height function*

$$H_{\mathcal{L}} : X(F) \longrightarrow \mathbb{R}_{>0}$$

on the set of F-rational points $X(F)$ (see 4.1 for more details). The set of F-rational points of height bounded by $B > 0$ is finite and we can define the *counting function*

$$N(U, \mathcal{L}, B) := \#\{x \in U(F) \mid H_{\mathcal{L}}(x) \leqslant B\},$$

where $U \subset X$ is a Zariski open subset.

Theorem 1.3.1. — *Let X/F be one of the following varieties:*

- *toric variety* [**5**]*;*
- *equivariant compactification of* $\mathbf{G}_a^n$ [**9**]*;*
- *flag variety* [**18**]*;*
- *equivariant compactification of* $\mathbf{G}/\mathbf{U}$ *- horospherical variety (where* $\mathbf{G}$ *is a semi-simple group and* $\mathbf{U} \subset \mathbf{G}$ *a maximal unipotent subgroup)* [**41**]*;*
- *smooth complete intersection of small degree (for example,* [**6**]*).*

Let $\mathcal{L}$ be an appropriate height on X such that the class $L \in \mathrm{Pic}(X)$ is contained in the interior of the cone of effective divisors.

Then there exists a dense Zariski open subset $U \subset X$ and constants

$$a(L), b(L), \Theta(U, \mathcal{L}) > 0$$

such that

$$N(U, \mathcal{L}, B) = \frac{\Theta(U, \mathcal{L})}{a(L)(b(L)-1)!} B^{a(L)} (\log(B))^{b(L)-1}(1 + o(1)),$$

as $B \to \infty$.

Remark 1.3.2. — The constants $a(L)$ and $b(L)$ depend only on the class of L in $\mathrm{Pic}(X)$. The constant $\Theta(U, \mathcal{L})$ depends, of course, not only on the geometric data (U, L) but also on the choice of the height. It is interpreted, in a general context, in [**5**].

Remark 1.3.3. — Notice that with the exception of complete intersections the varieties from Theorem 1.3.1 have a rather simple "cellular" structure. In particular, we can parametrize all rational points in some dense Zariski open subset. The theorem is to be understood as a statement about *heights*: even the torus $\mathbf{G}_m^2$ has very nontrivial embeddings into projective spaces and in each of these embeddings we have a different counting problem.

1.4. Techniques. — Let $\mathbf{G}$ be an algebraic torus or the group $\mathbf{G}_a^n$. The study of height asymptotics in these cases uses harmonic analysis on the adelic points $\mathbf{G}(\mathbb{A})$:

(1) Define a height pairing

$$H = \prod_v H_v \;:\; \mathrm{Pic}^{\mathbf{G}}(X)_{\mathbb{C}} \times \mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C},$$

(where $\mathrm{Pic}^{\mathbf{G}}(X)$ is the group of isomorphism classes of $\mathbf{G}$-linearized line bundles on X) such that its restriction to $L \in \mathrm{Pic}(X) \times \mathbf{G}(F)$ is the usual height $\mathcal{L}$ as before and such that H is invariant under a standard compact subgroup $\mathbf{K} \subset \mathbf{G}(\mathbb{A})$.

(2) Define the height zeta function

$$Z(\mathbf{G}, \boldsymbol{s}) = \sum_{x \in \mathbf{G}(F)} H(\boldsymbol{s}; x)^{-1}.$$

The projectivity of X implies that $Z(\mathbf{G}, \boldsymbol{s})$ converges for $\Re(\boldsymbol{s})$ in some (shifted) open cone in $\mathrm{Pic}^{\mathbf{G}}(X)_{\mathbb{R}}$.

(3) Apply the Poisson formula to obtain a representation

$$Z(\mathbf{G}, \boldsymbol{s}) = \int_{(\mathbf{G}(\mathbb{A})/\mathbf{G}(F)\mathbf{K})^*} \widehat{H}(\boldsymbol{s}; \chi) d\chi,$$

where the integral is over the group of unitary characters χ of $\mathbf{G}(\mathbb{A})$ which are trivial on $\mathbf{G}(F)\mathbf{K}$ and $d\chi$ is an appropriate Haar measure.

(4) Compute the Fourier transforms $\widehat{H}_v$ at almost all nonarchimedean places and find estimates at the remaining places.

(5) Prove a meromorphic continuation of $Z(\mathbf{G}, \boldsymbol{s})$ and identify the poles.
(6) Apply a Tauberian theorem.

2. Algebraic tori

For simplicity, we will always assume that $\mathbf{T}$ is a *split* algebraic torus over a number field F, that is, a connected reductive group isomorphic to $\mathbf{G}_{m,F}^d$, where $\mathbf{G}_{m,F} := \mathrm{Spec}(F[x, x^{-1}])$.

2.1. Adelization

***Notations 2.1.1* (Fields).** — Let F be a number field and $\mathrm{disc}(F)$ the discriminant of F (over $\mathbb{Q}$). The set of places of F will be denoted by $\mathrm{Val}(F)$. We shall write $v|\infty$ if v is archimedean and $v \nmid \infty$ if v is nonarchimedean. For any place v of F we denote by F_v the completion of F at v and by $\mathcal{O}_v$ the ring of v-adic integers (for $v \nmid \infty$). Let q_v be the cardinality of the residue field $\mathbb{F}_v$ of F_v for nonarchimedean valuations and put $q_v = e$ for archimedean valuations. The local absolute value $|\cdot|_v$ on F_v is the multiplier of the Haar measure, i.e., $d(ax_v) = |a|_v dx_v$ for some Haar measure dx_v on F_v. We denote by $\mathbb{A} = \mathbb{A}_F = \prod_v' F_v$ the adele ring of F.

***Notations 2.1.2* (Groups).** — Let $\mathbf{G}$ be a connected reductive algebraic group defined over a number field F. Denote by $\mathbf{G}(\mathbb{A})$ the adelic points of $\mathbf{G}$ and by

$$\mathbf{G}^1(\mathbb{A}) := \Big\{ g \in \mathbf{G}(\mathbb{A}) \mid \prod_{v \in \mathrm{Val}(F)} |m(g_v)|_v = 1 \ \ \forall m \in \widehat{\mathbf{G}}_F \Big\}$$

the kernel of F-rational characters $\widehat{\mathbf{G}}_F$ of $\mathbf{G}$.

***Notations 2.1.3* (Tori).** — Denote by $M = \widehat{\mathbf{T}}_F = \mathbb{Z}^d$ the group of F-rational characters of $\mathbf{T}$ and by $N = \mathrm{Hom}(M, \mathbb{Z})$ the dual group (as customary in toric geometry). Put $M_v := M$ (resp. $N_v := N$) for nonarchimedean valuations and $M_v := M \otimes \mathbb{R}$ for archimedean valuations.

Write $K_v \subset \mathbf{T}(F_v)$ for the maximal compact subgroup of $\mathbf{T}(F_v)$ (after fixing an integral model for $\mathbf{T}$ we have $K_v = \mathbf{T}(\mathcal{O}_v)$ for almost all v).

Choose a Haar measure $d\mu = \prod_v d\mu_v$ on $\mathbf{T}(\mathbb{A})$ normalized by $\mathrm{vol}(K_v) = 1$ (for nonarchimedean v the induced measure on $\mathbf{T}(F_v)/K_v$ is the discrete measure).

The adelic picture of a split torus $\mathbf{T}$ is as follows:
- $\mathbf{T}(\mathbb{A})/\mathbf{T}^1(\mathbb{A}) \simeq (\mathbf{G}_m(\mathbb{A})/\mathbf{G}_m^1(\mathbb{A}))^d \simeq \mathbb{R}^d$;
- $\mathbf{T}^1(\mathbb{A})/\mathbf{T}(F) = (\mathbf{G}_m^1(\mathbb{A})/\mathbf{G}_m(F))^d$ is compact;
- $\mathbf{K} = \prod_{v \in \mathrm{Val}(F)} K_v$;
- $\mathbf{T}^1(\mathbb{A})/\mathbf{T}(F)\mathbf{K}$ is a product of a finite group and a connected compact abelian group;
- $\mathbf{K} \cap \mathbf{T}(F)$ is a finite group of torsion elements.