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TORIC MORI THEORY AND FANO MANIFOLDS

by

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Abstract. — The following are the notes to five lectures on toric Mori theory and Fano manifolds given during the school on toric geometry which took place in Grenoble in Summer of 2000.

Contents

These are the notes to five lectures which I gave during the school on toric geometry in Grenoble in the Summer of 2000. The first week of the three week long school was meant to introduce the basics of toric geometry to the students while the other two weeks were devoted to advanced topics. Therefore the idea of the present notes is to give a brief and self-contained introduction to an advanced and broad topic to students who have just learned the fundamentals of toric language.

I claim no originality on the contents of these notes. Actually, they are primarily based on Miles Reid article [**11**]. An exposition of Mori theory in general can be found in [**5**]. Moreover Lecture 3 uses ideas of [**12**] while Lecture 5 is related to [**3**].

All varieties are algebraic and defined over $\mathbb{C}$.

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0. Short introduction: Minimal Model Program

In the course of the first week's lectures you have learned basics of toric geometry. You must have noticed that the theory is nice, clear and elegant, even too good to be true. And, indeed, that's right: toric varieties are very, very rare among algebraic varieties, so don't be confused: toric geometry is less than tip of the iceberg of algebraic geometry. Nevertheless it is very, very useful. Firstly because you can test your theories and conjectures (wisely posed!) in the toric environment. Secondly, because, as special as it is, toric geometry gives a very close insight in the local structure of varieties, where "local" is in analytic or formal neighborhood sense (not Zariski!). I'll try to illustrate these two principles in the course of my lectures.

We set for the classification of complex projective varieties of given dimension. Our primary examples are complex curves (or Riemann surfaces). The contents of the following table is referred to frequently when it comes to explaining principles of classification theory which includes the apparent trichotomy.

Sphere with g handles:	$g=0$	$g=1$	$g\geqslant 2$
Fundamental group:	trivial	$\mathbb{Z}^2$	$2g$ generators
Curvature:	positive	zero	negative
Holomorphic forms:	none	non-vanishing	g independent
Holomorphic vector fields:	2 independent	non-vanishing	none
Canonical divisor K_X:	negative	zero	positive

I shall focus on the canonical divisor. Let me recall the following:

Definition. — Let X be a normal variety of dimension n, with $X_0 \subset X$ denoting its smooth part. The canonical divisor K_X is a Weil divisor obtained by extending the divisor K_{X_0} associated to the sheaf of holomorphic n-forms $\Omega^n_{X_0} = \Lambda^n(\Omega_{X_0})$.

We will need moreover the following.

Definition. — Let L be a $\mathbb{Q}$-Cartier divisor on a normal variety X, that is, a multiple mL, with $m \in \mathbb{Z}$ is a Cartier divisor. We say that L is nef (numerically effective) if the intersection $L\cdot C = (1/m)\deg_C(mL_{|C})$ is non-negative for any compact curve $C \subset X$.

Apart from the curve case we have the following observation in dimension 2 which is an easy corollary to Enriques-Kodaira classification of smooth complex surfaces: each projective surface can be modified birationally – using blow-ups and blow-downs – either to a $\mathbb{P}^1$ bundle over a curve, or to a surface with nef canonical divisor.

Based on this (very roughly presented) evidence one can state

Minimal Model Conjecture. — *Any projective normal variety X is birationally equivalent to a normal projective variety X' which satisfies one of the following:*

(i) *X' admits Fano-Mori fibration $\varphi : X' \to Y$, that is: φ is a projective morphism with connected fibers, $\varphi_* \mathcal{O}_{X'} = \mathcal{O}_Y$, onto a normal variety Y, with* $\dim Y < \dim X'$, *and $-K_{X'}$ ample on fibers of φ, or*

(ii) *X' is minimal which means that $K_{X'}$ is nef (such X' is then called a minimal model of X).*

At this point I am rather vague about possible singularities of the involved varieties, however we have to assume that $K_{X'}$ is $\mathbb{Q}$-Cartier at least.

Here is an idea how to approach the Conjecture:

(1) Locate curves which have negative intersection with canonical divisor, understand their position in homology of X: use Cone Theorem [Mori, Kawamata].

(2) Eliminate some of these curves by contracting them to points: use Contraction Theorem [Kawamata, Shokurov]; chances are that we shall get Fano-Mori fibration, or we get a birational morphism to a simpler variety; unfortunately the birational map may also lead to a variety with very bad singularities (case of small contractions), so that the canonical divisor is not $\mathbb{Q}$-Cartier.

(3) If the contraction leads to bad singularities use birational surgery (flips) to replace curves which have negative intersection with K_X by curves which have positive intersection with K_X: this should be possible by Flip Conjecture (proved by Mori in dimension 3).

Although the Minimal Model Conjecture is void for toric varieties (they are rational, hence birational to a Fano-Mori fibration), they can be used effectively to test steps of the Program and possibly to describe local (in the analytic, or formal sense) geometry of non-minimal varieties. In the course of the present lectures I will review the main ideas of Minimal Model Program in the situation of toric varieties.

Let me recall toric notation.

$M \simeq \mathbb{Z}^n$	lattice of characters of a torus $T \simeq (\mathbb{C}^*)^n$
$N = \mathrm{Hom}(M, \mathbb{Z})$	lattice of 1-dimensional subgroups of T
$M_\mathbb{R}$ and $N_\mathbb{R}$	vector spaces in which they live
$\langle v_1, \ldots, v_k \rangle$	convex cone spanned on vectors $v_1, \ldots, v_k$
$\mathrm{conv}(v_1, \ldots, v_k)$	(affine) convex hull of points $v_1, \ldots, v_k$
$X = X(\Delta)$	toric variety associated to a fan Δ in $N_\mathbb{R}$
$\Delta(k)$	the set of k dimensional cones in Δ
$V(\sigma) \subset X(\Delta)$	stratum (= closure of the orbit) associated to a cone $\sigma \in \Delta$

Moreover, I will frequently confuse rays in $\Delta(1)$ with primitive elements from N generating them: for a ray $\rho \in \Delta(1)$ I will always consider the (unique) primitive element $e \in N \cap \rho$.

We usually assume that fans are *non-degenerate*, that is any cone $\sigma \in \Delta$ is strictly convex: $\sigma \cap -\sigma = 0$. Now I shall consider a slightly more general situation. Let

$V \subset N_R$ be a rational vector subspace, then I call Δ^* *a fan with vertex* V if it satisfies the usual conditions of a fan with strict convexity of cones replaced by

$$\forall\, \sigma \in \Delta^* : \sigma \cap -\sigma = V$$

The star * in Δ^* will indicate that the fan Δ^* has possibly non-trivial vertex. (The fans in the usual sense have vertices equal to $\{0\}$.)

If Δ^* is a fan in $N_{\mathbb{R}}$ with a vertex V then we can define a lattice $N' = N/(N \cap V)$, so that $N'_{\mathbb{R}} = N_{\mathbb{R}}/V$. Then the fan Δ^* descends to a nondegenerate fan Δ^*/V in $N'_{\mathbb{R}}$ and $X(\Delta^*/V)$ is a toric variety of dimension $n - \dim V$.

Let me recall that Γ is a sub-division of Δ if $|\Delta| = |\Gamma|$ and any cone in Δ is a union of cones from Γ. If both fans are non-degenerate then this defines a birational morphism $X(\Gamma) \to X(\Delta)$. If a fan Δ^* with a vertex V has a sub-division to a non-degenerate fan Γ then we have a morphism $X(\Gamma) \to X(\Delta^*/V)$, general fiber of which is of dimension $\dim V$.

1. Cone Theorem

First, let me recall basic facts about the intersection on toric varieties. We start with a complete algebraic variety X. Let $N^1(X) \subset H^2(X, \mathbb{R})$ and $N_1(X) \subset H_2(X, \mathbb{R})$ be the $\mathbb{R}$-linear subspaces spanned by, respectively, cohomology and homology classes of, respectively, Cartier divisors and holomorphic curves on X. The class of a curve C in $N_1(X)$ will be denoted by $[C]$.

The intersection of cycles and cocycles restricts to $N_1(X) \times N^1(X)$ and provides a non-degenerate pairing. Thus we can identify any space in question with the dual of its pairing partner.

The following definition describes a convenient class of varieties.

Definition. — A normal variety X is called $\mathbb{Q}$-factorial if some multiple of any Weil divisor is a Cartier divisor.

For toric varieties we have a clear description of $\mathbb{Q}$ factoriality.

Proposition. — *A toric variety $X = X(\Delta)$ is $\mathbb{Q}$-factorial if and only if the fan Δ is simplicial, that is all the cones in Δ are simplicial.*

Note that if $X = X(\Delta)$ is $\mathbb{Q}$-factorial then for any $\rho_i \in \Delta(1)$ the Weil divisor $V(\rho_i)$ is $\mathbb{Q}$-Cartier. Let $\mathbb{R}^{\Delta(1)}$ be an (abstract) real vector space in which vectors called $\widetilde{e}_i$, with e_i primitive in $\rho_i \in \Delta(1)$, form an orthonormal basis. We have the following exact sequences of vector spaces, dual each to the other,

$$0 \longrightarrow M_{\mathbb{R}} \longrightarrow \mathbb{R}^{\Delta(1)} \longrightarrow N^1(X) \longrightarrow 0$$

$$0 \longrightarrow N_1(X) \longrightarrow \mathbb{R}^{\Delta(1)} \longrightarrow N_{\mathbb{R}} \longrightarrow 0$$

with arrows in the first sequence defined as $M_{\mathbb{R}} \ni m \mapsto \sum e_i(m) \cdot \widetilde{e}_i$ and $\widetilde{e}_i \mapsto V(e_i)$ while the maps in the second sequence are as follows $N_1(X) \ni Z \mapsto \sum (Z \cdot V(\rho_i)) \cdot \widetilde{e}_i$ and $\widetilde{e}_i \mapsto e_i$.

Corollary. — *If $X = X(\Delta)$ is a $\mathbb{Q}$-factorial toric variety defined by a fan Δ then $N_1(X)$ can be interpreted as the space of linear relations between primitive vectors e_i in rays $\rho_i \in \Delta(1)$.*

Now, for an arbitrary variety X, we consider the following cones in the linear spaces defined above: the cone of curves (called also the cone of effective 1-cycles, or Mori cone) $NE(X) \subset N_1(X)$ and the cone of nef divisors $\mathcal{P} = \mathcal{P}(X) \subset N^1(X)$; they are $\mathbb{R}_{\geqslant 0}$-spanned by, respectively, the classes of curves and numerically effective divisors. Note that $\mathcal{P}$ and $\overline{NE}$ (the closure of NE) are — by their very definition — dual each to the other in the sense of the intersection pairing of $N^1(X)$ and $N_1(X)$. If X is projective then, by Kleiman criterion of ampleness, the cone $NE(X)$ is strictly convex.

Let me explain one of the starting points of the Program: Mori's move–bend–and–break argument. In toric case this is particularly explicit: if X is a complete toric variety then every effective cycle on X is numerically equivalent to a positive linear combination of some 1-dimensional strata of the big torus action.

Let $C \subset X(\Delta)$ be an irreducible curve. Suppose that C is contained in a stratum $V(\sigma)$ which is of the smallest dimension among the strata containing C. If $\dim V(\sigma) = 1$ then there is nothing to be done, otherwise we want to deform C to a union of curves belonging to lower-dimensional strata. We may assume — possibly by passing to a smaller dimensional toric variety — that $V(\sigma) = X(\Delta)$ which means that the general point of C is contained in the open orbit of $X(\Delta)$. If $\dim X(\Delta) = 2$ then we note that fixed points of the action of T on the linear system $|C|$ are associated to combination of 1-dimensional strata of $X(\Delta)$, hence we are done in this case.

Now, let $\dim X(\Delta) > 2$ and $C \subset X$ be an irreducible curve. Let $\lambda \in N$ be general and consider the action $\mathbb{C}^* \times X(\Delta) \to X(\Delta)$ of the 1-parameter group coming from λ, we denote it $(t, x) \mapsto t^{\lambda} \cdot x$. We may assume that the action has only a finite number of fixed points. The action gives a morphism $\mathbb{C}^* \times C \to X(\Delta)$ and hence a rational map $\mathbb{C} \times C - \to X(\Delta)$. Blowing up the points of indeterminacy we resolve this map, that is we find a surface S, a regular morphism $\psi : S \to X(\Delta)$ and a projection $\pi : S \to \mathbb{C}$, such that $\psi(\pi^{-1}(1)) = C$. Over $\mathbb{C}^* \times C$ we have a natural $\mathbb{C}^*$-action which lifts up to S so that both ψ and π are $\mathbb{C}^*$ equivariant. The (reducible) curve $\psi(\pi^{-1}(0))$ is invariant with respect to the action of λ, thus it is a union of closures of 1-dimensional orbits of λ. Note that to make it numerically equivalent to the original C the components of the curve $\psi(\pi^{-1}(0))$ may have to be assigned multiplicities depending on the degree of the map ψ on components of $\pi^{-1}(0)$; moreover, via the action of the group the generic point of C is moved toward a fixed point of the action (to so-called sink, or source, of the action on $X(\Delta)$) and thus the strict transform of $\{0\} \times C$ in S gets contracted to this point.