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SEMIGROUP ALGEBRAS AND DISCRETE GEOMETRY

by

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Abstract. — In these notes we study combinatorial and algebraic properties of affine semigroups and their algebras: (1) the existence of unimodular Hilbert triangulations and covers for normal affine semigroups, (2) the Cohen–Macaulay property and number of generators of divisorial ideals over normal semigroup algebras, and (3) graded automorphisms, retractions and homomorphisms of polytopal semigroup algebras.

Contents

1. Introduction

These notes, composed for the Summer School on Toric Geometry at Grenoble, June/July 2000, contain a major part of the joint work of the authors.

In Section 3 we study a problem that clearly belongs to the area of discrete geometry or, more precisely, to the combinatorics of finitely generated rational cones and their Hilbert bases. Our motivation in taking up this problem was the attempt

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to understand the normality of affine semigroups (and their algebras). The counter-example we have found shows that some natural conjectures on the structure of Hilbert bases do not hold, and that there is no hope to explain normality in terms of formally stronger properties. Nevertheless several questions remain open: for example, the positive results end in dimension 3, while the counter-examples live in dimension 6.

Section 4 can be viewed as an intermediate position between discrete geometry and semigroup algebras. Its objects are the sets T of solutions of linear diophantine systems of inequalities relative to the set S of solutions of the corresponding homogeneous systems: S is a normal semigroup and T can be viewed as a module over it. After linearization by coefficients from a field, the vector space KT represents a divisorial ideal over the normal domain $K[S]$ (at least under some assumptions on the system of inequalities). While certain invariants, like number of generators, can be understood combinatorially as well as algebraically, others, like depth, make sense only in the richer algebraic category.

The last part of the notes, Section 5, lives completely in the area of semigroup algebras. More precisely, its objects, namely the homomorphisms of polytopal semigroup algebras, can only be defined after the passage from semigroups to algebras. But there remains the question to what extent the homomorphisms can forget the combinatorial genesis of their domains and targets. As we will see, the automorphism groups of polytopal algebras have a perfect description in terms of combinatorial objects, and to some extent this is still true for retractions of polytopal algebras. We conclude the section with a conjecture about the structure of all homomorphisms of polytopal semigroup algebras.

Polytopal semigroup algebras are derived from lattice polytopes by a natural construction. While normal semigroup algebras in general, or rather their spectra, constitute the affine charts of toric varieties, the polytopal semigroup algebras arise as homogeneous coordinate rings of projective toric varieties. Several of our algebraic results can therefore easily be translated into geometric theorems about embedded projective toric varieties. Most notably this is the case for the description of the automorphism groups.

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2. Affine and polytopal semigroup algebras

2.1. Affine semigroup algebras. — We use the following notation: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are the additive groups of integral, rational, and real numbers, respectively; $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the corresponding additive subsemigroups of non-negative numbers, and $\mathbb{N} = \{1, 2, \ldots\}$.

Affine semigroups. — An *affine semigroup* is a semigroup (always containing a neutral element) which is finitely generated and can be embedded in $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Groups isomorphic to $\mathbb{Z}^n$ are called *lattices* in the following.

We write $\operatorname{gp}(S)$ for the group of differences of S, i. e. $\operatorname{gp}(S)$ is the smallest group (up to isomorphism) which contains S. Thus every element $x \in \operatorname{gp}(S)$ can be presented as $s - t$ for some $s, t \in S$.

If S is contained in the lattice L as a subsemigroup, then $x \in L$ is *integral* over S if $cx \in S$ for some $c \in \mathbb{N}$, and the set of all such x is the *integral closure* $\overline{S}_L$ of S in L. Obviously $\overline{S}_L$ is again a semigroup. As we shall see in Proposition 2.1.1, it is even an affine semigroup, and can be described in geometric terms.

By a *cone* in a real vector space $V = \mathbb{R}^n$ we mean a subset C such that C is closed under linear combinations with non-negative real coefficients. It is well-known that a cone is finitely generated if and only if it is the intersection of finitely many vector halfspaces. (Sometimes a set of the form $z + C$ will also be called a cone.) If C is generated by vectors with rational or, equivalently, integral components, then C is called *rational*. This is the case if and only if the halfspaces can be described by homogeneous linear inequalities with rational (or integral) coefficients.

This applies especially to the cone $C(S)$ generated by S in the real vector space $L \otimes \mathbb{R}$:

$$(*) \qquad C(S) = \{x \in L \otimes \mathbb{R} : \sigma_i(x) \geqslant 0,\ i = 1, \dots, s\}$$

where the σ_i are linear forms on $L \otimes \mathbb{R}$ with integral coefficients.

We consider a single halfspace

$$H_i = \{x \in L \otimes \mathbb{R} : \sigma_i(x) \geqslant 0\}.$$

The semigroup $L \cap H_i$ is isomorphic to $\mathbb{Z}_+ \oplus \mathbb{Z}^{n-1}$ where $n = \operatorname{rank} L$.

Note that the cone $C(S)$ is essentially independent of L. The embedding $S \subset L$ induces an embedding $\operatorname{gp}(S) \subset L$ and next an embedding $\operatorname{gp}(S) \otimes \mathbb{R} \subset L \otimes \mathbb{R}$.This embedding induces an isomorphism of the cones $C(S)$ formed with respect to $\operatorname{gp}(S)$ and L.

Proposition 2.1.1

(a) (Gordan's lemma) *Let $C \subset L \otimes \mathbb{R}$ be a finitely generated rational cone (i. e. generated by finitely many vectors from $L \otimes \mathbb{Q}$). Then $L \cap C$ is an affine semigroup and integrally closed in L.*

(b) *Let S be an affine subsemigroup of the lattice L. Then*
   (i) $\overline{S}_L = L \cap C(S)$;
   (ii) *there exist $z_1, \dots, z_u \in \overline{S}_L$ such that $\overline{S}_L = \bigcup_{i=1}^{u} z_i + S$;*
   (iii) *$\overline{S}_L$ is an affine semigroup.*

Proof. — (a) Note that C is generated by finitely many elements $x_1, \dots, x_m \in L$. Let $x \in L \cap C$. Then $x = a_1 x_1 + \cdots + a_m x_m$ with non-negative rational a_i. Set $b_i = \lfloor a_i \rfloor$.

Then

$$(*) \qquad x = (b_1x_1 + \cdots + b_mx_m) + (r_1x_1 + \cdots + r_mx_m), \qquad 0 \leqslant r_i < 1.$$

The second summand lies in the intersection of L with a bounded subset of C. Thus there are only finitely many choices for it. These elements together with $x_1, \dots, x_m$ generate $L \cap C$. That $L \cap C$ is integrally closed in L is evident.

(b) Set $C = C(S)$, and choose a system $x_1, \dots, x_m$ of generators of S. Then every $x \in L \cap C$ has a representation $(*)$. Multiplication by a common denominator of $r_1, \dots, r_m$ shows that $x \in \overline{S}_L$. On the other hand, $L \cap C$ is integrally closed by (a) so that $\overline{S}_L = L \cap C$.

The elements $y_1, \dots, y_u$ can now be chosen as the vectors $r_1x_1 + \cdots + r_mx_m$ appearing in $(*)$. There number is finite since they are all integral and contained in a bounded subset of $L \otimes \mathbb{R}$. Together with $x_1, \dots, x_m$ they certainly generate $\overline{S}_L$ as a semigroup. □

See Subsection 4.4 for further results on the finite generation of semigroups.

Proposition 2.1.1 shows that integrally closed affine semigroups can also be defined by finitely generated rational cones C: the semigroup $S(C) = L \cap C$ is affine and integrally closed in L.

We introduce special terminology in the case in which $L = \mathrm{gp}(S)$. Then the integral closure $\overline{S} = \overline{S}_{\mathrm{gp}(S)}$ is called the *normalization*, and S is *normal* if $S = \overline{S}$. Clearly the semigroups $S(C)$ are normal, and conversely, every normal affine semigroup S has such a representation, since $S = S(C(S))$ (in $\mathrm{gp}(S)$).

Suppose that $L = \mathrm{gp}(S)$ and that representation $(*)$ of $C(S)$ is irredundant. Then the linear forms σ_i describe exactly the support hyperplanes of $C(S)$, and are therefore uniquely determined up to a multiple by a non-negative factor. We can choose them to have coprime integral coefficients (with respect to $e_1 \otimes 1, \dots, e_r \otimes 1$ for some basis $e_1, \dots, e_r$ of $\mathrm{gp}(S)$), and then the σ_i are uniquely determined. We call them the *support forms* of S, and write

$$\mathrm{supp}(S) = \{\sigma_1, \dots, \sigma_s\}.$$

The map

$$\sigma : S \longrightarrow \mathbb{Z}^s, \qquad \sigma(x) = (\sigma_1(x), \dots, \sigma_s(x)),$$

is obviously a homomorphism that can be extended to $\mathrm{gp}(S)$. Obviously $\mathrm{Ker}(\sigma) \cap \overline{S}$ is the subgroup of $\overline{S}$ formed by its invertible elements: $x, -x \in C(S)$ if and only if $\sigma_i(x) = 0$ for all i.

Let $S_i = \{x \in S : \sigma_1(x) + \cdots + \sigma_s(x) = i\}$. Clearly $S = \bigcup_{i=0}^{\infty} S_i$, $S_i + S_j \subset S_{i+j}$ (and $S_0 = \mathrm{Ker}(\sigma) \cap S$). Thus σ induces a *grading* on S for which the S_i are the graded components. If we want to emphasize the graded structure on S, then we call $\sigma(x)$ the *total degree* of x.

We call a semigroup S *positive* if 0 is the only invertible element in S. It is easily seen that $\overline{S}$ is positive as well and that positivity is equivalent to the fact that $C(S)$

is a pointed cone with apex 0. Thus σ is an injective map, inducing an embedding $\overline{S} \to \mathbb{Z}_+^s$. We call it the *standard embedding* of $\overline{S}$ (or S).

One should note that a positive affine semigroup S can even be embedded into $\mathbb{Z}_+^r$, $r = \operatorname{rank}(S)$, such that the image generates $\mathbb{Z}_+^r$ as a group. We can assume that $\operatorname{gp}(S) = \mathbb{Z}^r$, and the dual cone

$$C(S)^* = \{\varphi \in (\mathbb{R}^r)^* : \varphi(x) \geqslant 0 \text{ for all } x \in S\}$$

contains r integral linear forms $\varphi_1, \dots, \varphi_r$ forming a basis of $(\mathbb{Z}^r)^*$ (a much stronger claim will be proved in Subsection 3.3). Then the automorphism $\Phi = (\varphi_1, \dots, \varphi_r)$ of $\mathbb{Z}^r$ yields the desired embedding. (The result is taken from [**Gu2**]; this paper discusses many aspects of affine semigroups and their algebras not covered by our notes).

If S is positive, then the graded components S_i are obviously finite. Moreover, every element of S can be written as the sum of irreducible elements, as follows by induction on the total degree. Since S is finitely generated, the set of irreducible elements is also finite. It constitutes the *Hilbert basis* $\operatorname{Hilb}(S)$ of S; clearly $\operatorname{Hilb}(S)$ is the uniquely determined minimal system of generators of S. For a cone C the Hilbert basis of $S(C)$ is denoted by $\operatorname{Hilb}(C)$ and called the *Hilbert basis* of C.

Especially for normal S the assumption that S is positive is not a severe restriction. In this case S_0 (notation as above) is the subgroup of invertible elements of S, and the normality of S forces S_0 to be a direct summand of S. Then the image S' of S under the natural epimorphism $\operatorname{gp}(S) \to \operatorname{gp}(S)/S_0$ is a positive normal semigroup. Thus we have a splitting

$$S = S_0 \oplus S'.$$

Semigroup algebras. — Now let K be a field. Then we can form the *semigroup algebra* $K[S]$. Since S is finitely generated as a semigroup, $K[S]$ is finitely generated as a K-algebra. When an embedding $S \to \mathbb{Z}^n$ is given, it induces an embedding $K[S] \to K[\mathbb{Z}^n]$, and upon the choice of a basis in $\mathbb{Z}^n$, the algebra $K[\mathbb{Z}^n]$ can be identified with the Laurent polynomial ring $K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Under this identification, $K[S]$ has the monomial basis X^a, $a \in S \subset \mathbb{Z}^n$ (where we use the notation $X^a = X_1^{a_1} \cdots X_n^{a_n}$).

If we identify S with the semigroup K-basis of $K[S]$, then there is a conflict of notation: addition in the semigroup turns into multiplication in the ring. The only way out would be to avoid this identification and always use the exponential notation as in the previous paragraph. However, this is often cumbersome. We can only ask the reader to always pay attention to the context.

It is now clear that affine semigroup algebras are nothing but subalgebras of $K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ generated by finitely many monomials. Nevertheless the abstract point of view has many advantages. When we consider the elements of S as members of $K[S]$, we will usually call them *monomials*. Products as with $a \in K$ and $s \in S$ are called *terms*.