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CONTINUOUS DIVISION OF LINEAR DIFFERENTIAL OPERATORS AND FAITHFUL FLATNESS OF $\mathscr{D}_X^\infty$ OVER $\mathscr{D}_X$

by

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Abstract. — In these notes we prove the faithful flatness of the sheaf of infinite order linear differential operators over the sheaf of finite order linear differential operators on a complex analytic manifold. We give the Mebkhout-Narváez's proof based on the continuity of the division of finite order differential operators with respect to a natural topology. We reproduce the proof of the continuity theorem given by Hauser-Narváez, which is simpler than the original proof.

***Résumé* (Continuité de la division des opérateurs différentiels et fidèle platitude de $\mathscr{D}_X^\infty$ sur $\mathscr{D}_X$)**

Dans ce cours on démontre la fidèle platitude du faisceau d'opérateurs différentiels linéaires d'ordre infini sur le faisceau d'opérateurs différentiels linéaires d'ordre fini d'une variéte analytique complexe lisse. La preuve que nous donnons est celle de Mebkhout-Narváez, qui utilise la continuité de la division d'opérateurs différentiels d'ordre fini par rapport à une topologie naturelle. Nous réproduisons la preuve de Hauser-Narváez du théorème de continuité, qui est plus simple que la preuve originale.

Introduction

The sheaf $\mathcal{O}_X$ of holomorphic functions on a complex analytic manifold X is the first natural example of left module over the sheaf of linear differential operators $\mathcal{D}_X$ on X. Here, as usual, differential operators have (locally) finite order. In fact, there is another natural sheaf of noncommutative rings extending $\mathcal{D}_X$, called the *sheaf of linear differential operators of infinite order*, $\mathcal{D}_X^\infty$, introduced by Sato. The left $\mathcal{D}_X$-module structure on $\mathcal{O}_X$ extends to a left $\mathcal{D}_X^\infty$-module structure in such a way that $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{O}_X = \mathcal{O}_X$.

For any holonomic left $\mathcal{D}_X$-module $\mathcal{M}$, we know by the constructibility theorem of Kashiwara [**9**] (see also [**12**], [**13**]) that the complex of holomorphic solutions of $\mathcal{M}$, $R\,Hom_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)$, is constructible. The canonical $\mathcal{D}_X$-linear biduality morphism

$$\mathcal{M} \to R\,Hom_{\mathbb{C}_X}(R\,Hom_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X),\mathcal{O}_X)$$

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induces a $\mathcal{D}_X^\infty$-linear morphism

$$(*)\qquad \mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M} \to R\,\mathcal{H}om_{\mathbb{C}_X}(R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X).$$

The *local biduality theorem* of Mebkhout asserts that (*) is an isomorphism for any holonomic module $\mathcal{M}$ (see [**11**, 11.3] in this volume). This theorem is an essential ingredient for the "full" Riemann-Hilbert correspondence, which establishes an equivalence between three categories: the bounded derived category of regular holonomic complexes of $\mathcal{D}_X$-modules, the bounded derived category of holonomic complexes of $\mathcal{D}_X^\infty$-modules and the bounded derived category of analytic constructible complexes (see 11.4 in loc. cit.). The sheaf $\mathcal{D}_X^\infty$ does not have any known finiteness properties like $\mathcal{D}_X$, but to prove the full Riemann-Hilbert correspondence one needs to know that the extension $\mathcal{D}_X \subset \mathcal{D}_X^\infty$ is faithfully flat. This result has been stated and proved for the first time in [**17**] (see also [**1**]), and its proof depended on the microlocal machinery.

The aim of these notes is to give an elementary self-contained proof of the faithful flatness of the sheaf of differential operators of infinite order over the sheaf of differential operators of finite order. The method we follow is that of [**14**], whose first step consists in considering the ring of differential operators of infinite order as the completion of the corresponding ring of finite order for a natural topology, and then mimic Serre's proof of the faithful flatness of the completion of a noetherian local ring over the ring itself [**18**]. The essential technical tool is the continuity of the Weierstrass-Grauert-Hironaka division of differential operators [**2, 3**]. We reproduce with detail the proof given in [**8**], which simplifies the original proof in [**14**]. As a complement we sketch the results of [**15**] for the case of differential operators with polynomial coefficients (Weyl algebra).

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1. Topological structure on rings of linear differential operators with analytic coefficients

Let X be a complex analytic manifold of pure dimension n, countable at infinity. Let us denote by $\mathcal{O}_X$ the sheaf of holomorphic functions and by $\mathcal{D}_X$ the sheaf of linear differential operators (*cf.* [**6**]). For each open set $U \subset X$, the space $\mathcal{O}_X(U)$ endowed with the topology of uniform convergence on compact sets is a *Fréchet space*, *i.e.* a complete metrizable locally convex space (it is also a *nuclear* space, *cf.* [**5**] for details). The Banach open mapping theorem shows that the property of being continuous for a $\mathbb{C}$-linear endomorphism $P : \mathcal{O}_X \to \mathcal{O}_X$ is a local property. For that, let $\{U_i\}$ be an open covering of X, that we can take as countable, such that each restriction $P|_{U_i} : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$ is continuous. For any open set $U \subset X$, the canonical injection $\mathcal{O}_X(U) \hookrightarrow \prod \mathcal{O}_X(U \cap U_i)$ is a closed inmersion by the open mapping theorem (its image is the kernel of the Čech map $\prod \mathcal{O}_X(U \cap U_i) \to \prod \mathcal{O}_X(U \cap U_i \cap U_j)$ by the sheaf condition). Hence, the continuity of $P(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ comes from the continuity of $\prod P(U \cap U_i) : \prod \mathcal{O}_X(U \cap U_i) \to \prod \mathcal{O}_X(U \cap U_i)$. As a consequence, the pre-sheaf of $\mathbb{C}$-linear continuous endomorphisms of $\mathcal{O}_X$, $\mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$, is actually a sheaf.

The following proposition is well-known (*cf.* [**14**], prop. 2.1.4):

Proposition 1.1. — *For any continuous $\mathbb{C}$-linear endomorphism $P : \mathcal{O}_X \to \mathcal{O}_X$ and for any system $(U; x_1, \ldots, x_n)$ of local coordinates of X, there are unique holomorphic functions $a_\alpha \in \mathcal{O}_X(U)$, $\alpha \in \mathbb{N}^n$, such that*

$$P|_U = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{1}{\alpha!} \partial^\alpha,$$

with $\partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $\lim_{|\alpha| \to \infty} |a_\alpha|^{1/|\alpha|} = 0$ uniformly on any compact set of U. Equivalently, the function

$$(p, \xi) \in U \times \mathbb{C}^n \longmapsto \sum_{\alpha \in \mathbb{N}^n} a_\alpha(p) \xi^\alpha \in \mathbb{C}$$

is holomorphic.

From now on, we will denote $\mathcal{D}_X^\infty = \mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$ and call it *sheaf of infinite order linear differential operators*. From the above proposition we deduce that it coincides with the sheaf of infinite order linear differential operators defined in [**16, 17**].

The following proposition is proved in [**14**], prop. 2.1.3.

Proposition 1.2. — *Let $P : \mathcal{O}_X \to \mathcal{O}_X$ be a $\mathbb{C}$-linear endomorphism. The following properties are equivalent:*

a) *P is continuous.*

b) *For any pair $K, K' \subseteq X$ of compact sets with $K \subset \mathring{K}'$, there is a constant $C_{K,K'} > 0$ such that $|P(f)|_K \leqslant C_{K,K'} |f|_{K'}$ for any holomorphic function f defined on a neighborhood of K'.*

Corollary 1.3. — *The sheaf $\mathcal{D}_X$ of (finite order) linear differential operators is a subsheaf (of rings) of $\mathit{Homtop}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof. — Let P be a section of $\mathcal{D}_X$ over an open set $U \subset X$. Since continuity is a local property, we can suppose that U is a connected open set of $\mathbb{C}^n$. Then P admits a unique expression

$$P = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant d} a_\alpha \frac{1}{\alpha!} \partial^\alpha,$$

where d is the order of P and the a_α are holomorphic functions on U. Let $K, K' \subseteq U$ be a pair of compact sets as in proposition 1.2, b) and let f be a holomorphic function on a neighborhood of K'. From Cauchy inequalities we deduce that

$$|P(f)|_K = \Big| \sum_{|\alpha|\leqslant d} a_\alpha \frac{1}{\alpha!}\partial^\alpha(f)\Big|_K \leqslant \sum_{|\alpha|\leqslant d} |a_\alpha|_K r^{-|\alpha|}|f|_{K'}$$

where r is the distance between K and $U - \overset{\circ}{K'}$. By proposition 1.2, we conclude that P is continuous. □

Definition 1.4 **([14], déf. 2.1.6)**. — For any open set $U \subseteq X$, the *canonical topology* of $\mathcal{D}_X^\infty(U)$ or $\mathcal{D}_X(U)$ is defined as the locally convex topology given by the semi-norms

$$p_{(K,K')} : P \in \mathcal{D}_X^\infty(U) \longmapsto p_{(K,K')}(P) := \sup\{|P(f)|_K/|f|_{K'} \mid f \in \mathcal{O}_X(K'), f \neq 0\},$$

indexed by pairs (K, K') of compact sets in U with $K \subset \overset{\circ}{K'}$.

For any coordinate system $(U; x_1, \ldots, x_n)$ in X, we can use Cauchy inequalities as in corollary 1.3 and proposition 1.1 to prove that the map

$$\sum_{\alpha\in\mathbb{N}^n} a_\alpha(\underline{x})\frac{1}{\alpha!}\partial^\alpha \longmapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha(\underline{x})\underline{y}^\alpha$$

is an isomorphism of locally convex vector spaces between $\mathcal{D}_X^\infty(U)$ endowed with the canonical topology and the space of holomorphic functions on $U \times \mathbb{C}^n$ endowed with the topology of uniform convergence on compact sets. This isomorphism depends on local coordinates and carries the space $\mathcal{D}_X(U)$ into the space of holomorphic functions on $U \times \mathbb{C}^n$ which are polynomials with respect to the second factor. Consequently, $\mathcal{D}_X^\infty(U)$ is a Fréchet (and nuclear) space and $\mathcal{D}_X(U)$ is dense in $\mathcal{D}_X^\infty(U)$. We can write then $\mathcal{D}_X^\infty(U) = \widehat{\mathcal{D}_X(U)}$.

In fact, in [**14**, §2] it is proved that $\mathcal{D}_X^\infty$ endowed with the canonical topology is a sheaf with values in the category of Fréchet $\mathbb{C}$-algebras.

Let us denote by $\mathcal{O}_n, \mathcal{D}_n, \mathcal{D}_n^\infty$ the stalk at the origin of the sheaves $\mathcal{O}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n}, \mathcal{D}_{\mathbb{C}^n}^\infty$ respectively. For $\rho = (\rho_1, \ldots, \rho_n)$, $L = (L_1, \ldots, L_n)$ in $(\mathbb{R}_+^*)^n$ let us consider the pseudo-norm $|-|_\rho^L : \mathcal{D}_n^\infty \to \mathbb{R}_+ \cup \{+\infty\}$ whose value at $P = \sum a_\beta \partial^\beta = \sum_{\alpha\beta} a_{\alpha\beta} x^\alpha \partial^\beta$ is

$$(1) \qquad |P|_\rho^L = \sum_\beta |a_\beta|_\rho |\beta|! L^\beta = \sum_{\alpha\beta} |a_{\alpha\beta}| \cdot |\beta|! \rho^\alpha L^\beta \in \mathbb{R}_+ \cup \{+\infty\}.$$

Since $\beta! \leqslant |\beta|! \leqslant n^{|\beta|}\beta!$, we could also use $\beta!$ instead $|\beta|!$ in (1) to obtain an equivalent system of pseudo-norms. Nevertheless, the choice of $|\beta|!$ is forced by the proofs of the majorations needed to obtain the norm estimates of theorem 2.11 (see [**14**, 2.2.4] and [**8**]).

Let us denote by $\mathcal{D}_n^\infty(\rho)$ the subspace of $\mathcal{D}_n^\infty$ where $|-|_\rho^L$ takes finite values for any $L \in (\mathbb{R}_+^*)^n$ and let us write $\mathcal{D}_n(\rho) := \mathcal{D}_n \cap \mathcal{D}_n^\infty(\rho)$. The semi-norms $|-|_\rho^L$, $L \in (\mathbb{R}_+^*)^n$, define a Fréchet topology on $\mathcal{D}_n^\infty(\rho)$.

Following [**8**], we consider weights $\lambda, \mu \in (\mathbb{N}^*)^n$ and, for real numbers $s, t > 0$, $\rho = s^\lambda = (s^{\lambda_1}, \ldots, s^{\lambda_n})$, $L = t^{-\mu} = (t^{-\mu_1}, \ldots, t^{-\mu_n})$. When λ is fixed, we denote $|-|_s^{\mu,t} := |-|_\rho^L$, $\mathcal{D}_n^\infty(s) := \mathcal{D}_n^\infty(\rho)$ and $\mathcal{D}_n(s) := \mathcal{D}_n(\rho)$.

In the case where U is an open polycylinder of $\mathbb{C}^n$ centered at 0 of polyradius $\sigma = s_0^\lambda$, $0 < s_0 \leqslant +\infty$, we have

$$\mathcal{D}_{\mathbb{C}^n}^\infty(U) = \bigcap_{0<s<s_0} \mathcal{D}_n^\infty(s), \quad \mathcal{D}_{\mathbb{C}^n}(U) = \bigcap_{0<s<s_0} \mathcal{D}_n(s),$$

and the canonical topology of $\mathcal{D}_{\mathbb{C}^n}^\infty(U)$ (resp. $\mathcal{D}_{\mathbb{C}^n}(U)$) is the (topological) inverse limit of the $\mathcal{D}_n^\infty(s)$ (resp. $\mathcal{D}_n(s)$), for $0 < s < s_0$. In other words, the canonical topologies of $\mathcal{D}_{\mathbb{C}^n}^\infty(U)$ and $\mathcal{D}_{\mathbb{C}^n}(U)$ are given by the semi-norms $|-|_s^{\mu,t}$, $0 < s < s_0$, $t^{-\mu} \gg 0$ [**14**], 2.2.3. The last condition can be obtained with μ fixed and $t \to 0$, or taking $t = t(s) < 1$ and $\mu \gg 0$.

For vectors $P = (P_1, \ldots, P_q) \in (\mathcal{D}_n^\infty)^q$, following [**7**] we also define

$$|P|_s^{\mu,t} := \sum_{i=1}^q |P_i|_s^{\mu,t} s^{-(i-1)},$$

where $\lambda \in (\mathbb{N}^*)^n$ is fixed.

In the above situation, the product topology on $\mathcal{D}_{\mathbb{C}^n}^\infty(U)^q$ and $\mathcal{D}_{\mathbb{C}^n}(U)^q$ is also given by the semi-norms $|-|_s^{\mu,t}$, $0 < s < s_0$, $t^{-\mu} \gg 0$.

2. The continuity theorem

In this section, we fix $M_1, \ldots, M_r \in \mathcal{D}_n^q$ and a total well ordering $<$ in $\mathbb{N}^{2n}$ compatible with sums (*cf.* [**4**, 1.3]). Whenever we speak about the ordering $<$ in $\mathbb{N}^{2n} \times \{1, \ldots, q\}$ we mean the ordering induced by $<$ in the following way:

$$(\alpha, \beta, i) < (\alpha', \beta', j) \Longleftrightarrow \begin{cases} (\alpha, \beta) < (\alpha', \beta') \\ \text{or} \\ (\alpha, \beta) = (\alpha', \beta') \text{ and } i > j \end{cases}$$

Given

$$N = (N_1, \ldots, N_q) = \sum_{i=1}^q N_i e_i = \sum_{i=1}^q \sum_{\alpha,\beta} a_{\alpha\beta i} x^\alpha \partial^\beta e_i \in \mathcal{D}_n^q, \quad a_{\alpha\beta i} \in \mathbb{C},$$

where $\{e_i\}_{i=1\ldots q}$ stands for the canonical basis of $\mathcal{D}_n^q$ as a free $\mathcal{D}_n$-module, we denote by $\mathscr{N}(N)$, the *Newton diagram* of N, the set of (α, β, i) in $\mathbb{N}^{2n} \times \{1, \ldots, q\}$ such that $a_{\alpha\beta i} \neq 0$ and by $\sigma(N)$ its *symbol*, *i.e.* the homogeneous component of N of maximal degree with respect to the grading given by the total degree in ∂:

$$\sigma(N) = \sum_{i=1}^q \sum_{|\beta|=d} \sum_\alpha a_{\alpha\beta i} x^\alpha \partial^\beta e_i, \quad d = \deg_{\mathrm{T}}(N) = \max \deg(N_i).$$