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GEOMETRIC IRREGULARITY AND $\mathscr{D}$-MODULES

by

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***Abstract.* —** In the one dimensional case, J.-P. Ramis associated a Newton polygon to an analytic differential operator. On this polygon may be read the irregularity of the operator as well as its indices in various functional spaces. This result is here generalized in the higher dimensional case. We define a Newton polygon and positive microcharacteristic cycles. We get so a purely algebraic definition of the characteristic cycle of the irregularity of a holonomic $\mathscr{D}$-module.

***Résumé* (Irrégularité géométrique et $\mathscr{D}$-modules). —** En une variable, J.-P. Ramis a associé à un opérateur différentiel analytique un polygone de Newton sur lequel on peut lire l'irrégularité de cet opérateur ainsi que ses indices dans divers espaces fonctionnels. On montre ici que ce résultat se généralise en dimension quelconque, en définissant un polygone de Newton et des cycles microcaractéristiques positifs. En particulier, on obtient une définition purement algébrique du cycle caractéristique de l'irrégularité d'un $\mathscr{D}$-module holonome.

Introduction

Let X be a complex manifold and $\mathscr{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Regular holonomic $\mathscr{D}_X$-modules are completely determined by the Riemann-Hilbert correspondence which is an equivalence of categories between these modules and the perverse sheaves on X. In the non regular case, things are much more complicated.

When the dimension of X is 1, the irregularity of an ordinary differential equation is just a positive number. In higher dimensions, it may be understood as a perverse sheaf as explained by Mebkhout in this Summer School or as a positive cycle as we will see here. The relation between these two points of view is simply the fact that the positive cycle is the characteristic cycle associated to the perverse sheaf. But in fact, the two methods are completely different and give complementary results.

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As shown in dimension 1 by the results of Ramis, the irregularity itself is not sufficient and we have to define a finite family of positive cycles, not only one. This will be done by a method which is very similar to the definition of the characteristic cycle. We will define a family of filtrations on the sheaf $\mathscr{D}_X$ and from it we will get the family of microcharacteristic cycles.

More precisely, if Y is a submanifold of X, the microcharacteristic cycles form a finite family of lagrangian cycles on the space $T^*T^*_YX$ (cotangent to the conormal space to Y). They give a formula to compute the index of solutions of the $\mathscr{D}_X$-module. In particular they compute the index of the sheaf of irregularity introduced by Mebkhout.

But these cycles are not the good ones. Let us assume that Y is a hypersurface. Then the sheaf of irregularity is a sheaf on Y and we need cycles on T^*Y not on $T^*T^*_YX$. We show that to each lagrangian cycle on $T^*T^*_YX$ with a suitable action of $\mathbb{C}^*$ is associated a cycle on T^*Y, called its irregularity, which has good properties of homogeneity and positivity.

Applying this to $\mathscr{D}_X$-modules, we get positive cycles on T^*Y which compute the index of the sheaf of irregularity and which vanishes if and only if $\mathscr{M}$ is regular along Y. Moreover we show that these cycles are positive (positivity of the irregularity) and that they are divisible by an integer (the denominator of the slope). These properties generalize the properties of the irregularity in dimension one. In particular it generalizes the positivity while the last property is the generalization of the fact that the vertices of Newton Polygon have integral coordinates.

The detailed proofs are not given here but may be found in [**5**] and [**6**].

1. Ordinary differential equations

1.1. Newton Polygon (*cf.* Ramis [9]). — Let X be an open neighborhood of 0 in $\mathbb{C}$ and and P a differential operator on X:

$$P(t, D_t) = \sum_{0 \leqslant j \leqslant m} p_j(t) D_t^j$$

(with $D_t^j = d^j/dt^j$). Developing the p_j functions in Taylor series near 0 we get:

$$P(t, D_t) = \sum_{\substack{0 \leqslant j \leqslant m \\ i \geqslant 0}} p_{ij} t^i D_t^j$$

For $0 \leqslant j \leqslant m$, we denote by k_j the valuation of the function p_j at 0 (i.e. the highest power of t dividing p_j) and we define:

$$S_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu \geqslant k_j - j \}$$

Then $S_0(P)$ is the union of the sets S_j and the Newton Polygon $\mathbb{N}_0(P)$ is the convex hull of $S_0(P)$. It is a convex subset of $\mathbb{R}^2$ (Figure 1).

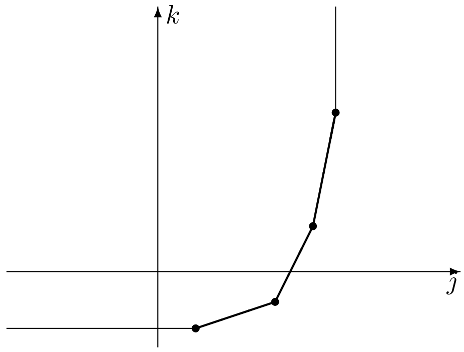

FIGURE 1. Newton Polygon

The operator P is said to be regular at 0, or to have "regular singularities" if the Newton Polygon has only one vertex.

In the general case, this polygon is made of two half-lines (one vertical, one horizontal) and of a finite number of segments. We denote by $0 < s_N < \cdots < s_1 < +\infty$ the slopes of these segments and by $1 < r_1 < \cdots < r_N < +\infty$ the rational numbers given by $(r_i - 1)s_i = 1$. The numbers r_i are, by definition, the slopes of P or the "algebraic slopes" of P (sometimes also called the critical indexes of P).

The sum $\sum p_{ij} t^i \tau^j$ over $(j, i-j)$ on the vertical half-line of the Newton Polygon is nothing else than the function $p_m(t)\tau^m$ where m is the order of P that is the principal symbol of P. In a similar way, we define the the determining equation of P relative to the index r as the restriction to $t = 1$ of the sum $\sum p_{ij} t^i \tau^j$ over $(j, i-j)$ on the segment of slope $1/(r-1)$.

If r is not a slope of P, the corresponding determining equation is monomial, otherwise it is a polynomial function of τ. The Newton Polygon is determined up to a translation by the list of the degrees and valuations of the determining equations.

1.2. The algebraic case. — If all the coefficients of P are polynomial in t, we may define a "negative part" of the Newton Polygon. Keeping the previous notations, we denote by d_j the degree of p_j and replace the sets S_j by the two families:

$$S'_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu = k_j - j \} \tag{1.2.1}$$

$$S''_j = \{ (\lambda, \mu) \in \mathbb{R}^2 \mid \lambda \leqslant j,\ \mu = d_j - j \} \tag{1.2.2}$$

We get a Newton Polygon with positive and negative slopes (Figure 2).

1.3. Formal power series. — When P is regular at 0, Fuchs theorem says that all formal power series which are solutions of the equation $Pu = 0$ are convergent.

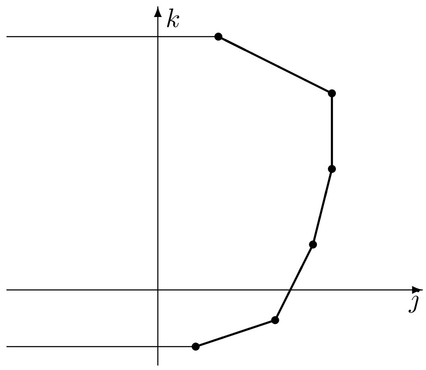

FIGURE 2. Second Newton Polygon

In [**7**], Malgrange has defined the irregularity of P as

$$\operatorname{Irr}(P) = \chi(P, \mathbb{C}[[t]]) - \chi(P, \mathbb{C}\{t\})$$

where $\mathbb{C}[[t]]$ is the ring of formal power series and $\mathbb{C}\{t\}$ the ring of convergent series.

Let us recall that if P is an operator on a $\mathbb{C}$-vector space F, P has finite index if the kernel and the cokernel of P are finite dimensional $\mathbb{C}$-vector spaces and the index $\chi(P, F)$ of P is:

$$\chi(P, F) = \dim_{\mathbb{C}} \operatorname{Ker}(P) - \dim_{\mathbb{C}} \operatorname{Coker}(P)$$

Malgrange proved that the irregularity is equal to the height between the higher and the lower vertex of the Newton Polygon of P (with the definition of section 1.1). It is thus a positive number which vanishes if and only if P is regular.

Ramis [**9**] improved this results using the rings $\mathbb{C}[[t]]_r$ of Gevrey formal powers. A formal power series $u(t) = \sum_{k\geqslant 0} u_k t^k$ is an element of $\mathbb{C}[[t]]_r$ if and only if:

$$F_r[u](t) := \sum_{k\geqslant 0} u_k \frac{t^k}{(k!)^{r-1}}$$

is convergent.

***Theorem 1.3.1* (Ramis [9])**

(1) *The operator P has a finite index on $\mathbb{C}[[t]]_r$ for any $r \geqslant 1$.*

(2) *If u is a formal power series solution of the equation $Pu = 0$, it belongs to one of the spaces $\mathbb{C}[[t]]_r$ where r is a slope of P and the convergence radius of $F_r[u]$ is equal to the inverse of the modulus of one of the roots of the corresponding determining equation.*

(3) *The index $\chi(P, \mathbb{C}[[t]]_r)$, as a function of r, is constant outside of the points r which are slopes of P. Its jump at one of these points is equal to the height of the segment of slope $1/(r-1)$ of the Newton Polygon of P.*

1.4. Holomorphic microfunctions. — The previous result may be stated and proved in other families of functions. We will consider in particular the family of holomorphic microfunctions with support in $\{0\}$ which is easier to generalize in higher dimensional case as we shall see later.

If U is a neighborhood of 0 in $\mathbb{C}$ the quotient $\mathscr{O}(U-\{0\})\,/\mathscr{O}(U)$ does not depend on U, it is denoted by $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$. The subspace generated by meromorphic functions at 0 is denoted by $\mathscr{B}_{\{0\}|\mathbb{C}}$.

The space $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ operates on germs of holomorphic functions at 0 by the Cauchy formula. If f is a holomorphic function on a neighborhood U of 0 and if $u\in\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is represented by a function $\varphi(t)$ on $U-\{0\}$, we choose a path γ in $U-\{0\}$ such that the index of 0 is 1, e.g. a small circle centered at 0 and we set:

$$\langle u,f\rangle=\int_{\gamma}\varphi(t)f(t)dt$$

In this way, the class of the function $\frac{1}{2i\pi}\frac{1}{t}$ is identified to the Dirac operator δ : $f\mapsto f(0)$ and the function

$$\Phi_k(t)=\frac{(-1)^{k+1}}{2i\pi}\frac{k!}{t^{k+1}}$$

to the k-th derivative $\delta^{(k)}(t)$. This shows that an element of $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is written in a unique way:

$$u(t)=\sum_{k\in\mathbb{N}}a_k\delta^{(k)}(t)$$

where a_k is a sequence of complex numbers satisfying:

$$\forall\,\varepsilon>0,\,\exists\,C_\varepsilon>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C_\varepsilon\varepsilon^k\frac{1}{k!}$$

and such an u is an element of $\mathscr{B}_{\{0\}|\mathbb{C}}$ if and only if the sum is finite. $\mathscr{B}_{\{0\}|\mathbb{C}}$ is thus the space of distributions with support $\{0\}$ and $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ the space of hyperfunctions with support $\{0\}$.

For $r\geqslant 1$, we define the spaces of ultradistributions $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ and $\mathscr{B}_{\{0\}|\mathbb{C}^{(r)}}$. An element $u(t)=\sum a_k\delta^{(k)}(t)$ of $\mathscr{B}^{\infty}_{\{0\}|\mathbb{C}}$ is in $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ if the sequence a_k satisfies:

$$\forall\,\varepsilon>0,\,\exists\,C_\varepsilon>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C_\varepsilon\varepsilon^k\frac{1}{(k!)^r}$$

and it is in $\mathscr{B}_{\{0\}|\mathbb{C}^{(r)}}$ if

$$\exists\,C>0,\,\forall\,k\geqslant 0,\quad |a_k|\leqslant C^{k+1}\frac{1}{(k!)^r}$$

The spaces $\mathbb{C}[[t]]_r$ et $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ carry natural topologies for which they are topologically duals and the theorem 1.3.1 may be translated to $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$.

***Theorem 1.4.1* (Ramis [9])**

(1) *The operator P has a finite index on $\mathscr{B}_{\{0\}|\mathbb{C}\{r\}}$ and $\mathscr{B}_{\{0\}|\mathbb{C}^{(r)}}$ for any $r\geqslant 1$.*