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THE LOCAL DUALITY THEOREM IN $\mathscr{D}$-MODULE THEORY

by

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Abstract. — These notes are devoted to the Local Duality Theorem for $\mathscr{D}$-modules, which asserts that the topological Grothendieck-Verdier duality exchanges the de Rham complex and the solution complex of holonomic modules over a complex analytic manifold. We give Mebkhout's original proof and the relationship with Kashiwara-Kawai's proof. In that way we are able to precise the commutativity of some diagrams appearing in the last one.

***Résumé* (Le théorème de dualité locale dans la théorie des $\mathscr{D}$-modules).** — Ce cours est consacré au théorème de du alité locale pour les $\mathscr{D}$-modules, qui affirme que la dualité topologique de Grothendieck-Verdier échange le complexe de de Rham et le complexe des solutions des modules holonomes sur une variété analytique complexe. On donne la preuve originale de Mebkhout en faisant le rapport avec la preuve de Kashiwara-Kawai. Ceci nous permet de préciser la commutativité de certains diagrammes dans cette dernière.

Contents

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Introduction

These notes are issued from a course taught in the C.I.M.P.A. School on Differential Systems, held at Seville (Spain) from September 2 through September 13, 1996. They are an improved version of the handwritten notes distributed during the School.

The aim of these notes is to introduce the reader to the Local Duality Theorem in $\mathcal{D}$-module Theory —LDT for short— and to explain in a detailed way the proofs of it in [**Me3**], [**K-K**]. This theorem asserts that the Verdier duality for analytic constructible complexes interchanges the "De Rham" and the "Solutions" of every bounded holonomic complex of $\mathcal{D}$-modules on a complex manifold. Besides the importance and the beauty of such a result, it is a good representative of the relationship between discrete and continuous coefficients, an important idea in contemporary Algebraic Geometry.

The first published duality type result is a punctual one due to Kashiwara [**Ka**], § 5. The LDT in the way we currently use was first stated by Mebkhout in [**Me1**], 4.1, [**Me2**], 5.2, but its proof depended on a still conjectural theory of Topological Homological Algebra. A complete proof was given in [**Me3**], III.1.1 (see also [**Me4**], 1.1, [**Me5**], ch. I, 4.3). Kashiwara and Kawai proposed another proof in [**K-K**], 1.4.6 based on the punctual result above.

The proof of the punctual result of Kashiwara uses the Local Duality in Analytic Geometry (residues). Mebkhout's proof of the LDT uses Serre and Poincaré-Verdier dualities to construct the duality morphism and to prove it is an isomorphism. Kashiwara and Kawai define the duality morphism as the formal one and reduce the proof of the LDT to the former result of Kashiwara by means of the Biduality Theorem for analytic constructible complexes. However, this reduction demands the commutativity of some diagram involving the global formal duality morphism and the punctual one, which is not obvious. Both proofs are evidently based on the Kashiwara's Constructibility Theorem.

In these notes we prove that the duality morphism defined by Mebkhout coincides with the formal one and, as a consequence, that the diagram needed in Kashiwara-Kawai's proof is commutative. This fact is explained by the relationship between the Global Serre Duality and the Local Duality in Analytic Geometry (*cf.* [**Li**]).

As we could expect, to do the task we need to be especially attentive to the definition and the properties of the different formal objects involved. In particular, we have to manage some signs. A complete reference for these questions is [**De2**], 1.1. For the sake of completeness and for the ease of the reader, we have collected (a big portion of) them in the Appendix.

Other somewhat different proofs of the LDT are available in [**Bo2**], § 19, [**Sa**], 2.7, [**Bj**], III, 3.3.10. We have chosen to present the first proof of the LDT, due to Mebkhout, and the proof of Kashiwara-Kawai because they are conceptually simple and they fit in this collective work as a continuation of [**M-S1, M-S2**].

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Notation

Given a sheaf of rings $\mathcal{R}_X$ on a topological space X, we shall denote by $C^*(\mathcal{R}_X), K^*(\mathcal{R}_X)$ and $D^*(\mathcal{R}_X)$ the category of complexes, the homotopy category of complexes and the derived category of the abelian category of left $\mathcal{R}_X$-modules respectively. We shall use ${}_r\mathcal{R}_X$ for referying to the category of right $\mathcal{R}_X$-modules.

The symbols $\mathcal{A}^\bullet, \mathcal{B}^\bullet, \mathcal{C}^\bullet$, etc. will be used for complexes of sheaves on a topological space: the objects of $\mathcal{A}^\bullet$ are the $\mathcal{A}^n$ and the differentials are $d^n_{\mathcal{A}} : \mathcal{A}^n \to \mathcal{A}^{n+1}$, for every $n \in \mathbb{Z}$.

Given a complex $\mathcal{A}^\bullet$ and an integer d, we shall denote by $h^d(\mathcal{A}^\bullet)$ its dth cohomology object.

Given a complex $A^\bullet$ (of objects in some additive category), the complex $A^\bullet[1]$ is defined by $A^\bullet[1]^n = A^{n+1}$, $d_{A^\bullet[1]} = -d_A$.

The total derived functors of $\operatorname{Hom}^\bullet_{\mathcal{R}_X}(-,-), \mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$ and $- \overset{\bullet}{\otimes}_{\mathcal{R}_X} -$ will be denoted by $\mathbf{R}\operatorname{Hom}^\bullet_{\mathcal{R}_X}(-,-), \mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$ and of $- \overset{\bullet}{\otimes}_{\mathcal{R}_X} -$ respectively, and $\mathcal{E}xt^d_{\mathcal{R}_X}(-,-) = h^d\,\mathbf{R}\,\mathcal{H}om^\bullet_{\mathcal{R}_X}(-,-)$.

If $\mathcal{R}_X$ is the constant sheaf associated to a fixed ring K and no confusion is possible, we shall abreviate $\operatorname{Hom}^\bullet_{K_X}(-,-)$, $\mathcal{H}om^\bullet_{K_X}(-,-)$, $\mathbf{R}\,\mathcal{H}om^\bullet_{K_X}(-,-)$ and $\mathcal{E}xt^d_{K_X}(-,-)$ by $\operatorname{Hom}^\bullet_X(-,-)$, $\mathcal{H}om^\bullet_X(-,-)$, $\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-)$ and $\mathcal{E}xt^d_X(-,-)$ respectively.

1. Duality for Analytic Constructible Sheaves

Throughout this section X denotes a connected complex analytic manifold countable at infinity of dimension d, and $D^b_c(\mathbb{C}_X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces with analytic constructible cohomology (*cf.* [**Ve**], [**Ka**], [**M-N3**]). We denote $\mathbb{T}_X = \mathbb{C}_X[2d]$.

1.1. The Topological Biduality Morphism. — The abelian category of sheaves of complex vector spaces over X has finite injective dimension (*cf.* [**DP**], exp. 2, 4.3). The functor $\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-)$ induces a functor

$$\mathbf{R}\,\mathcal{H}om^\bullet_X(-,-) : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \longrightarrow D^b(\mathbb{C}_X)$$

which can be computed by taking injective resolutions of the second argument, or locally free resolutions of the first argument if they exist.

1.1.1. Proposition. — *If $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$ are two complexes in $D_c^b(\mathbb{C}_X)$, then $\mathbf{R}\,\mathit{Hom}_X^\bullet(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ is also in $D_c^b(\mathbb{C}_X)$. Furthermore, if the $\mathcal{F}_i^\bullet$ are constructible with respect to a Whitney stratification Σ of X, then $\mathbf{R}\,\mathit{Hom}_X^\bullet(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ is also constructible with respect to Σ.*

Proof[(1)]. — We can suppose that the $\mathcal{F}_i^\bullet$ are single constructible sheaves $\mathcal{F}_i$ (*cf.* [**M-N3**], II.5). The question being local (*cf. loc. cit.*, I.4.21) we can suppose that $\mathcal{F}_1 = \sigma_! \mathcal{L}$, for $\sigma : S \hookrightarrow X$ the inclusion of a stratum of Σ and $\mathcal{L}$ a local system (of finite rank) on S (*cf. loc. cit.*, I.4.14). In this case we have $\mathbf{R}\,\mathit{Hom}_X^\bullet(\sigma_!\mathcal{L}, \mathcal{F}_2) \simeq \mathbf{R}\,\sigma_* \mathbf{R}\,\mathit{Hom}_S^\bullet(\mathcal{L}, \sigma^! \mathcal{F}_2)$, and we can conclude by induction on the dimension of X and Thom-Whitney's isotopy theorem (*cf. loc. cit.*, I.4.15). □

1.1.2. Definition. — For every bounded complex $\mathcal{F}^\bullet$ in $D_c^b(\mathbb{C}_X)$ we define its *dual* by

$$\mathcal{F}^{\bullet\vee} := \mathbf{R}\,\mathit{Hom}_X^\bullet(\mathcal{F}^\bullet, \mathbb{C}_X)$$

and the *topological biduality morphism* $\boldsymbol{\beta}_{\mathcal{F}^\bullet} : \mathcal{F}^\bullet \to (\mathcal{F}^{\bullet\vee})^\vee$ as in A.2.

1.1.3. Proposition. — *If $\mathcal{F}^\bullet$ is a bounded constructible complex on X, then for each point $x \in X$ and for every small ball B centered in x with respect to some local coordinates, the complex $\mathbf{R}\,\Gamma_c(B, \mathcal{F}^\bullet)$ has finite dimensional cohomology.*

Proof. — According to proposition 1.1.1, the complex $\mathcal{F}^{\bullet\vee}$ is bounded and constructible. Then, for every small ball B centered in x, the canonical morphism $\mathbf{R}\,\Gamma(B, \mathcal{F}^{\bullet\vee}) \to (\mathcal{F}^{\bullet\vee})_x$ is an isomorphism (*cf.* [**M-N3**], I.4.16) and we conclude by the Poincaré-Verdier duality

$$\mathbf{R}\,\Gamma(B, \mathcal{F}^{\bullet\vee}) = \mathbf{R}\,\mathrm{Hom}_B^\bullet(\mathcal{F}^\bullet|_B, \mathbb{C}_B) \xrightarrow{\simeq} \mathbf{R}\,\mathrm{Hom}_{\mathbb{C}}^\bullet(\mathbf{R}\,\Gamma_c(B, \mathcal{F}^\bullet), \mathbb{C})[-2d]$$

(*cf.* [**DP**], exp. 5). □

1.2. The Biduality Theorem. — The Biduality Theorem for analytic constructible sheaves has been first stated and proved by Verdier in [**Ve**], 6.2 using Resolution of Singularities. Other proofs in the setting of *cohomologically constructible sheaves* are available in [**DP**], exp. 10, § 2, [**Bo1**], V, 8.10, [**K-S**], 3.4. We sketch here a proof following the lines in [**SGA 4$\frac{1}{2}$**], Th. finitude, 4.3 and [**M-N3**], III.2.1,III.2.6 and based on the Poincaré-Verdier duality *cf.* [**DP**], exp. 4,5, [**Bo1**], V, 7.17, [**Iv**], VII.5.2, [**K-S**], 3.1.10.

1.2.1. Theorem. — *For each bounded constructible complex $\mathcal{F}^\bullet$ on X, the biduality morphism $\boldsymbol{\beta}_{\mathcal{F}^\bullet} : \mathcal{F}^\bullet \to (\mathcal{F}^{\bullet\vee})^\vee$ is an isomorphism.*

Proof. — We can suppose that $\mathcal{F}^\bullet$ is a single constructible sheaf $\mathcal{F}$ (*cf.* [**M-N3**], II.5). The result is clear if $\mathcal{F}$ is a local system (of finite rank).

(1) This proof is also valid in the case of an arbitrary complex analytic space.

As the question is local, we can also suppose that $X = D_1^{d-1} \times D_2$, where the D_i are open disks in $\mathbb{C}$, $\mathcal{F}$ is a local system on the complement of an hypersurface $Z \subset X$ and the first projection $p\colon X \to D_1^{d-1}$ is finite over Z (*cf. loc. cit.*, I.4.20).

We can extend our data, first to a constructible sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X} = D_1^{d-1} \times \mathbb{C}$ and second to $\overline{\mathcal{F}} = \sigma_! \widetilde{\mathcal{F}}$, where $\sigma : \widetilde{X} \hookrightarrow \overline{X} = D_1^{d-1} \times \mathbb{P}_1$ is the (open) inclusion. Call $\overline{p} : \overline{X} \to Y = D_1^{d-1}$ the first projection, which is proper.

Let us consider the triangle

$$\overline{\mathcal{F}} \xrightarrow{\boldsymbol{\beta}_{\overline{\mathcal{F}}}} (\overline{\mathcal{F}}^{\vee})^{\vee} \longrightarrow \mathcal{Q}^{\bullet} \longrightarrow \overline{\mathcal{F}}[1] \tag{1}$$

where the support of the (bounded) complex $\mathcal{Q}^{\bullet}$ is contained in $Z \cup (Y \times \{\infty\})$ and then it is finite over Y.

By taking direct images by $\overline{p}$ we obtain a new triangle in $D_c^b(\mathbb{C}_Y)$

$$\mathbf{R}\,\overline{p}_* \overline{\mathcal{F}} \xrightarrow{\mathbf{R}\,\overline{p}_* \boldsymbol{\beta}_{\overline{\mathcal{F}}}} \mathbf{R}\,\overline{p}_* (\overline{\mathcal{F}}^{\vee})^{\vee} \longrightarrow \mathbf{R}\,\overline{p}_* \mathcal{Q}^{\bullet} \longrightarrow \mathbf{R}\,\overline{p}_* \overline{\mathcal{F}}[1]$$

(*cf.* [**M-N3**], I.4.23).

In order to prove that $\boldsymbol{\beta}_{\overline{\mathcal{F}}}$ is an isomorphism we need to prove that $\mathcal{Q}^{\bullet} = 0$, but that is equivalent to $\mathbf{R}\,\overline{p}_* \mathcal{Q}^{\bullet} = 0$ because $\overline{p}$ is finite over the support of $\mathcal{Q}^{\bullet}$.

Let $\mathrm{Tr}_{X/Y} : \mathbf{R}\,\overline{p}_* \mathbb{T}_{\overline{X}} \to \mathbb{T}_Y$ be the *topological trace morphism* for the proper map $\overline{p}$. According to the local form of the Poincaré-Verdier duality (*cf.* [**Iv**], VII.5, [**K-S**], 3.1.10) the morphism $\boldsymbol{\rho}_{\mathcal{K}^{\bullet}}$ composition of

$$\begin{aligned} \mathbf{R}\,\overline{p}_*\, \mathbf{R}\, \mathit{Hom}^{\bullet}_{\overline{X}}(\mathcal{K}^{\bullet}, \mathbb{T}_{\overline{X}}) &\xrightarrow{\text{nat.}} \mathbf{R}\, \mathit{Hom}^{\bullet}_{Y}(\mathbf{R}\,\overline{p}_* \mathcal{K}^{\bullet}, \mathbf{R}\,\overline{p}_* \mathbb{T}_{\overline{X}}) \\ &\xrightarrow{(\mathrm{Tr}_{X/Y})_*} \mathbf{R}\, \mathit{Hom}^{\bullet}_{Y}(\mathbf{R}\,\overline{p}_* \mathcal{K}^{\bullet}, \mathbb{T}_{Y}) \end{aligned}$$

is an isomorphism for every bounded complex of sheaves of $\mathbb{C}$-vector spaces $\mathcal{K}^{\bullet}$.

Call $\boldsymbol{\rho}^*_{\overline{\mathcal{F}}} := \mathbf{R}\, \mathit{Hom}^{\bullet}_{Y}(\boldsymbol{\rho}_{\overline{\mathcal{F}}}, \mathbb{T}_Y)$ the isomorphism induced by $\boldsymbol{\rho}_{\overline{\mathcal{F}}}$. According to A.5, we can "redefine"

$$(\overline{\mathcal{F}}^{\vee})^{\vee} = \mathbf{R}\, \mathit{Hom}^{\bullet}_{\overline{X}}(\mathbf{R}\, \mathit{Hom}^{\bullet}_{\overline{X}}(\overline{\mathcal{F}}, \mathbb{T}_{\overline{X}}), \mathbb{T}_{\overline{X}})$$

and using A.2 and lemma A.15 we deduce the relation

$$\Big(\boldsymbol{\rho}_{\mathbf{R}\, \mathit{Hom}^{\bullet}_{\overline{X}}(\overline{\mathcal{F}}, \mathbb{T}_{\overline{X}})}\Big) \circ \mathbf{R}\,\overline{p}_* \boldsymbol{\beta}_{\overline{\mathcal{F}}} = \boldsymbol{\rho}^*_{\overline{\mathcal{F}}} \circ \boldsymbol{\beta}_{\mathbf{R}\,\overline{p}_* \overline{\mathcal{F}}}.$$

By induction hypothesis, the morphism $\boldsymbol{\beta}_{\mathbf{R}\,\overline{p}_* \overline{\mathcal{F}}}$ is an isomorphism, then $\mathbf{R}\,\overline{p}_* \boldsymbol{\beta}_{\overline{\mathcal{F}}}$ too and we obtain the desired $\mathbf{R}\,\overline{p}_* \mathcal{Q}^{\bullet} = 0$. □

1.2.2. As X is a connected oriented manifold of (topological) dimension $2d$, the *topological trace morphism* $\mathrm{tr}_X : \mathrm{H}^{2d}_c(X, \mathbb{C}_X) \to \mathbb{C}$ given by integration of top C^{∞}-forms with compact support is an isomorphism. Then, for each point $x \in X$, denoting by $i : \{x\} \hookrightarrow X$ the inclusion, the canonical morphism $i^! \mathbb{C}_X \to \mathbf{R}\,\Gamma_c(X, \mathbb{C}_X)$ gives rise to a *punctual topological trace* isomorphism

$$\mathrm{tr}_x : \mathrm{H}^{2d}_x(\mathbb{C}_X) \xrightarrow[\simeq]{\text{nat.}} \mathrm{H}^{2d}_c(X, \mathbb{C}_X) \xrightarrow[\simeq]{\mathrm{tr}_X} \mathbb{C}.$$