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EXPLICIT CALCULATIONS IN RINGS OF DIFFERENTIAL OPERATORS

by

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Abstract. — We use the notion of a standard basis to study algebras of linear differential operators and finite type modules over these algebras. We consider the polynomial and the holomorphic cases as well as the formal case.
Our aim is to demonstrate how to calculate classical invariants of germs of coherent (left) modules over the sheaf $\mathscr{D}$ of linear differential operators over $\mathbb{C}^n$. The main invariants we deal with are: the characteristic variety, its dimension and the multiplicity of this variety at a point of the cotangent space.
In the final chapter we shall study more refined invariants of $\mathscr{D}$-modules linked to the question of irregularity: The slopes of a $\mathscr{D}$-module along a smooth hypersurface of the base space.

Résumé **(Calculs explicites dans l'anneau des opérateurs différentiels).** — Dans ce cours on développe la notion de base standard, en vue d'étudier les algèbres d'opérateurs différentiels linéaires et les modules de type fini sur ces algèbres. On considère le cas des coefficients polynomiaux, des coefficients holomorphes ainsi que le cas des algèbres d'opérateurs à coefficients formels.
Notre but est de montrer comment les bases standards permettent de calculer certains invariants classiques des germes de modules (à gauche) cohérents sur le faisceaux $\mathscr{D}$ des opérateurs différentiels linéaires sur $\mathbb{C}^n$. Les principaux invariants que nous examinons sont : la variété caractéristique, sa dimension et sa multiplicité en un point du fibré cotangent.
Dans le dernier chapitre nous étudions des invariants plus fins des $\mathscr{D}$-modules qui sont reliés aux questions d'irrégularité : les pentes d'un $\mathscr{D}$-module, le long d'une hypersurface lisse.

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Introduction

The purpose of these notes is to make an account of explicit methods, using the notion of a standard basis, which could be used in studying algebras of linear differential operators and finite type modules over these algebras. We consider in parallel each of the following cases: coefficients in a ring of polynomials $\boldsymbol{k}[x_1, \ldots, x_n]$ for the Weyl algebra $A_n(\boldsymbol{k})$, in the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$ for $\mathscr{D}_n$, or in the ring of formal power series for $\widehat{\mathscr{D}_n}$. We denote $\mathscr{R}$ any of these rings of operators and $\mathscr{B}$ the corresponding commutative ring of coefficients.

Our aim is to demonstrate how to calculate classical invariants of germs of coherent (left) modules over the sheaf $\mathscr{D}$ of linear differential operators over $\mathbb{C}^n$. In practice we shall look at finite type modules over $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$. The main invariants we are dealing with are: the characteristic variety, and the multiplicity of this variety at a point of the cotangent space. See [**25**] and [**19**] for an introduction to the theory of $\mathscr{D}$-modules and for the definition of the characteristic variety, of its dimension and and of its multiplicity. In the last chapter we shall study more refined invariants of $\mathscr{R}$-modules linked to the question of irregularity: The slopes of a $\mathscr{D}_n$-module or an $A_n(\boldsymbol{k})$-module along a smooth hypersurface of the base space. In these notes we deal mainly with the case of monogenic modules $\mathscr{M} = \mathscr{R}/I$ with I a (left) ideal of $\mathscr{R}$. We provide an algorithm to build standard bases of I and in the context of chapter II these bases yield a special kind of system of generators for which the module of relations is easy to describe. There is a straightforward generalisation for the case $\mathscr{M} = \mathscr{R}^p/\mathscr{N}$ involving a submodule $\mathscr{N}$ of $\mathscr{R}^p$. Then continuing the process of building standard bases for submodules we can thus obtain a (locally) free resolution of $\mathscr{M}$. The techniques used are the notion of privileged exponents with respect to an ordering and a theorem of division. They were introduced by H. Hironaka (*cf.* [**26**] or [**1**]). In the polynomial case the notion of a standard basis was developed by Buchberger under the name of a Gröbner basis in [**13**] where he also gives an algorithm for its calculation.

The commutative case is treated in chapter I, where we recall the notions of a privileged exponent of a polynomial or a power series with respect to a convenient ordering, the definition of a standard basis and the algorithm for calculating it, which is the Buchberger's algorithm in the polynomial case. We also draw attention to the elegant proof in the convergent case taken from Hauser and Muller (*cf.* [**20**].) We finish by giving some applications in commutative algebra such as calculating multiplicities, syzygies, and the intersections of ideals.

In chapter II, we consider division processes in algebras of operators which are compatible with a filtration which may either be the filtration by the order of operators or in the particular case of $A_n(\boldsymbol{k})$, the Bernstein filtration by the total order. At the same time, for the sake of completeness we treat a weighted homogeneous version of these filtrations. Using a compatible ordering on monomials we again develop a division algorithm and an algorithm for the construction of a standard basis. These algorithms are very similar to those developed in chapter I, since in fact a division by a family of operators $\{P_1, \dots, P_r\}$, or by a standard basis of an ideal I induces the same object via the principal symbols in the commutative associated graded rings. The references for these results are [**11**] and [**14**]. Let us also notice that it is only in the case of $\boldsymbol{k}[x_1, \dots, x_r]$ or $A_n(\boldsymbol{k})$ that the suitable orderings used in chapters I and II are well orderings and therefore that the algorithms are effective. In the power series case they depend on formal or convergent processes in the local rings of series.

In chapter III we give an algorithm for the calculation of the slopes of a coherent $\mathscr{R}$-module along a smooth hypersurface Y of $\boldsymbol{k}^n$ or $\mathbb{C}^n$ in the neighbourhood of a point of Y. The material is essentially taken from our work with A.Assi [**2**] where however only the case of $A_n(\boldsymbol{k})$ is considered.

The notion of a slope of a coherent $\mathscr{D}$-module $\mathscr{M}$ was introduced by Y. Laurent under the name of a critical index. He considers, in the more general context of microdifferential operators a family of filtrations $L_r = pF + qV$ (with r a rational number such that $0 \leqslant r = p/q \leqslant +\infty$), which is an interpolation between the filtration by the order F and the V-filtration of Malgrange and Kashiwara (*cf.* [**22**]). The critical indices are those for which the L_r-characteristic variety of $\mathscr{M}$ is not bihomogeneous with respect to F and V. Laurent proved in loc. cit. the finiteness of the number of slopes and then C. Sabbah and F. Castro proved the same result in [**30**] by using a local flattener. In [**28**] Z. Mebkhout introduced the notion of a transcendental slope of a holonomic $\mathscr{D}$-module $\mathscr{M}$, as being a jump in the Gevrey filtration $\mathrm{Irr}_Y^{(r)}(\mathscr{M})$ of the irregularity sheaf $\mathrm{Irr}_Y(\mathscr{M})$. The irregularity sheaf is the complex of solutions of $\mathscr{M}$ with values in the quotient of the formal completion along Y of the structural sheaf $\mathscr{O}$, by $\mathscr{O}$ itself. By the main result of [**28**], it is a perverse sheaf, and $\mathrm{Irr}_Y^{(r)}(\mathscr{M})$ is the sub-perverse sheaf of solutions in formal series of Gevrey type r along Y. In [**23**] Laurent and Z. Mebkhout proved that the transcendental slopes of an holonomic $\mathscr{D}$-module are equal to the slopes in the sense of Laurent called algebraic slopes. The analogue in dimension one is Malgrange's paper [**27**] for the perversity of the irregularity sheaf and Ramis's paper [**29**] for the theorem of the comparison of slopes.

In chapter III, we recall the principle of the algorithm of calculation of the algebraic slopes of an $\mathscr{R}$-module that we developed in [**2**] and we give some supplementary information. Here the additional difficulty is that the linear form L_r which yields the similarly called filtration now possesses a negative coefficient in the variable x_1. Although we can still speak of privileged exponents and standard bases, the standard bases are no longer systems of generators of the ideal I which we consider but only induce a standard basis of the graded associated ideal. A more serious consequence of non-positivity, is that the straightforward division algorithm does not work inside *finite order* operators. The way to solve this problem is to homogenize the operators in $\mathscr{R}[t]$ with respect to the order filtration or, in the case of $A_n(\boldsymbol{k})$, with respect to the Bernstein filtration. We notice in chapter III, following a remark made by L. Narváez

[**16**] that we can simplify the original proof in [**2**] by considering on $A_n[t]$ a different structure as a Rees ring. Another improvement to [**2**] lies in the distinction between the slopes in the sense of Laurent and the values of r for which the ideal I gives a non-bihomogeneous graded ideal $\mathrm{gr}_{L_r}(I)$. We call those r, the idealistic slopes of I. In [**2**] we considered only this set of slopes and proved its finiteness; this paper however already contains the hard part of the algorithm of the calculation of algebraic slopes. Let us end this introduction by pointing out two other extensions of the original material of our paper [**2**]. First we make the same algorithm work for the rings of operators $\mathscr{D}_n$, or $\widehat{\mathscr{D}_n}$. Secondly we give some significant examples of the calculations of slopes: the slopes of the direct image of $\mathscr{D}_{\mathbb{C}}e^{1/x^k}$ by an immersion in $\mathbb{C}^2$, with respect to a smooth curve Y tangent to the support. This example contains idealistic slopes which end up not being algebraic slopes. Finally, we calculate the slopes of $\mathscr{D}_{\mathbb{C}^2}e^{1/(y^p-x^q)}$ along any line through the origin.

Added on March 21, 2003. — This paper was written in September 1996, as material for a six hour course given in the CIMPA summer school "Differential Systems" (Sevilla, September 1996). Consequently, the bibliography is outdated. Since then, many papers have been published about the computational aspects in $\mathscr{D}$-modules theory. We have therefore decided to add, after the references, a complementary list of recent publications on the subject.

1. Division theorems in polynomial rings and in power series rings

1.1. Let $\boldsymbol{k}$ be a field, with an arbitrary characteristic unless otherwise stated. Let n be a positive integer . We denote by:

- $\boldsymbol{k}[\boldsymbol{X}] = \boldsymbol{k}[X_1, \ldots, X_n]$ the ring of polynomials with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$.
- $\boldsymbol{k}[[\boldsymbol{X}]] = \boldsymbol{k}[[X_1, \ldots, X_n]]$ the ring of formal power series with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$.
- $\boldsymbol{k}\{\boldsymbol{X}\} = \boldsymbol{k}\{X_1, \ldots, X_n\}$ the ring of convergent power series with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$, if $\boldsymbol{k} = \mathbb{R}$ or $\mathbb{C}$. (1)

If $f \in \boldsymbol{k}[[\boldsymbol{X}]]$, $f \neq 0$, we write $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \boldsymbol{X}^\alpha$ where $f_\alpha \in \boldsymbol{k}$. If $f \in \boldsymbol{k}[\boldsymbol{X}]$ $f \neq 0$, then this sum is finite. The set $\mathscr{N}(f) = \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}$ is called the Newton diagram of the power series or of the polynomial f.

1.2. L-degree and L-valuation. — Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form with non negative coefficients.

Definition 1.2.1. — Let $0 \neq f \in \boldsymbol{k}[\boldsymbol{X}]$. We define the L-degree of f (and we denote it by $\deg_L(f)$) as being $\max\{L(\alpha) \mid f_\alpha \neq 0\}$. We set $\deg_L(0) = -\infty$.

(1) Or, more generally, a complete valued field.

Definition 1.2.2. — Let $0 \neq f \in \boldsymbol{k}[[\boldsymbol{X}]]$. We define the L-valuation of f (which we denote by $\mathrm{val}_L(f)$) as being $\min\{L(\alpha) \mid f_\alpha \neq 0\}$. We set $\mathrm{val}_L(0) = +\infty$.

We have $\deg_L(fg) = \deg_L(f) + \deg_L(g)$ if $f, g \in \boldsymbol{k}[\boldsymbol{X}]$ and $\mathrm{val}_L(fg) = \mathrm{val}_L(f) + \mathrm{val}_L(g)$ if $f, g \in \boldsymbol{k}[[\boldsymbol{X}]]$.

Definition 1.2.3. — Let $0 \neq f \in \boldsymbol{k}[[\boldsymbol{X}]]$. We call the sum $\mathrm{in}_L(f) = \sum_{L(\alpha)=\mathrm{val}_L(f)} f_\alpha \boldsymbol{X}^\alpha$ the L- initial form of the power series f [(2)]. Let I be an ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$. We call the ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$ generated by $\{\mathrm{in}_L(f) \mid f \in I\}$, the initial ideal of I . We denote it by $\mathrm{In}_L(I)$ (or simply $\mathrm{In}(I)$)

Notation. — The following notation will be useful. If $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha$ is a power series, we set $\mathrm{in}_{L,\nu}(f) = \sum_{L(\alpha)=\nu} f_\alpha \boldsymbol{X}^\alpha$. When no confusion can occur, we write $\mathrm{in}_\nu(f)$ instead of $\mathrm{in}_{L,\nu}(f)$. We have: $f = \sum_\nu \mathrm{in}_\nu(f)$.

Definition 1.2.4. — Let $0 \neq f \in \boldsymbol{k}[\boldsymbol{X}]$. We call the sum $\mathrm{fin}_L(f) = \sum_{L(\alpha)=\deg_L(f)} f_\alpha \boldsymbol{X}^\alpha$ the L-final form of the polynomial f. Let I be an ideal of $\boldsymbol{k}[\boldsymbol{X}]$. We call the ideal of $\boldsymbol{k}[\boldsymbol{X}]$ generated by $\{\mathrm{fin}_L(f) \mid f \in I\}$ the final ideal of I. We denote it by $\mathrm{Fin}_L(I)$ (or simply by $\mathrm{Fin}(I)$).

1.3. Orderings in $\mathbb{N}^n$. — Let $<$ be a total well ordering on $\mathbb{N}^n$ compatible with sums (*i.e.* if $\alpha, \beta \in \mathbb{N}^n$ and $\alpha < \beta$ then we have $\alpha + \gamma < \beta + \gamma$ for any $\gamma \in \mathbb{N}^n$). Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form with non negative coefficients . The relation $<_L$, defined by:

$$\alpha <_L \beta \text{ if and only if } \begin{cases} L(\alpha) < L(\beta) \\ \text{or } L(\alpha) = L(\beta) \text{ and } \alpha < \beta \end{cases}$$

is a total well ordering on $\mathbb{N}^n$ compatible with sums.

1.4. The privileged exponent of a polynomial or of a power series. — The notion of the privileged exponent of a power series is due to H. Hironaka. It was introduced in [**26**] (see also [**1**], [**10**]). We fix, once and for all, a total well ordering $<$, compatible with sums, in $\mathbb{N}^n$. Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form as above.

Definition 1.4.1. — Let $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha \in \boldsymbol{k}[\boldsymbol{X}], f \neq 0$. We call:

- The n-uple $\exp_L(f) = \max_{<_L}\{\alpha \mid f_\alpha \neq 0\}$, the L-privileged exponent of f
- The monomial $\mathrm{mp}_L = f_{\exp_L(f)} \boldsymbol{X}^{\exp_L(f)}$, the L-privileged monomial of f

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(2) If all the coefficients of L are positive, then the initial form of a power series is a polynomial.