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ASYMPTOTIC SOLUTIONS OF NON LINEAR WAVE EQUATIONS AND POLARIZED NULL CONDITIONS

by

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Dédié à la mémoire de Jean Leray,
un mathématicien exceptionnel et un grand homme.

Abstract. — The jump in generality made by Leray for the WKB type construction of high frequency asymptotic solutions of linear partial differential equations has allowed the treatment of arbitrary linear systems of partial differential equations. It also permitted the extension to quasilinear systems, and the appearance of new properties linked to the non linearities, in particular a distorsion of signals. The non linearity of a differential system is also an obstruction to the existence of global solutions of evolution problems. In the case of non linear wave equations on the Minkowski spacetime of dimension 4 it has been discovered by Christodoulou and Klainerman that a "null condition" satisfied by the non linearities leads to global existence results. The equations of the fundamental field equations (standard model, Einstein equations) are quasi linear second order partial differential equations, but not well posed due to gauge invariance. We introduce a "polarized null condition". We show it is satisfied by the standard model, but not quite by the Einstein equations. We construct for both systems asymptotic high frequency solutions with linear transport law along the rays. In the case of Einstein equations the wave inflicts a "back reaction" on the background metric.

***Résumé* (Conditions nulles polarisées).** — La généralisation faite par Leray de la méthode WKB pour la construction de solutions asymptotiques à haute fréquence de systèmes arbitraires d'équations aux dérivées partielles linéaires a permis le traitement de systémes quasilinéaires et l'apparition de propriétés nouvelles comme la distorsion des signaux. La non linéarité est aussi une obstruction à l'existence de solutions globales des systèmes d'évolution. On introduit une condition nulle polarisée, généralisation de la condition nulle de Christodoulou-Klainerman à des systèmes mal posés par suite de l'invariance de jauge. On montre qu'elle conduit à une équation de transport linéaire le long des rayons d'une solution asymptotique. Elle est satisfaite par le modèle standard, mais un terme résiduel dans le cas des équations d'Einstein conduit à une « réaction en retour » sur la métrique de base.

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1. Introduction

Leray [11], and Gårding Kotake Leray [7] have brought a fundamental improvement to the WKB construction of high frequency asymptotic solutions of linear partial differential equations as functions of the form $u = ve^{i\omega\varphi}$, with v a slowly varying amplitude, ω a large parameter and φ a scalar function called the phase. The method had be extended by Lax [10] to the construction of asymptotic solutions of first order linear systems as formal series

$$u = e^{i\omega\varphi}(v_0 + \frac{1}{\omega}v_1 + \cdots).$$

The jump in generality made by Leray allowed the treatment of arbitrary linear systems of partial differential equations. It also permitted the extension to quasilinear systems [2], and the appearance of new properties linked to the non linearities in some sense similar to shocks(1), in particular a distorsion of signals. The non linearity of a differential system is also an obstruction to the existence of global solutions of evolution problems. In the case of non linear wave equations on the Minkowski spacetime of dimension 4 it has been discovered by Christodoulou [6] and Klainerman [8] that a null condition satisfied by the non linearities leads to global existence results. The equations of the fundamental field equations (standard model, Einstein equations) are quasi linear second order partial differential equations, but not well posed due to gauge invariance. We introduce a polarized "null condition". We show it is satisfied by the standard model, but not quite by the Einstein equations. We construct for both these systems asymptotic high frequency solutions with linear transport law along the rays. In the case of Einstein equations the wave inflicts a "back reaction" [4] on the background metric, as was already noticed in[3].

2. The GKL linear theory

2.1. Linear systems. — We change slightly the notations of GKL to give it the geometrical aspects that it does possess. We write a linear differential system on a smooth pseudo riemannian manifold V under the form

$$L(x, D)u = b(x)$$

with x a point of V of local coordinates x^α, D the covariant derivative and u a field on V. The system reads in local coordinates and index notation

$$L^A_B(x, D)u^B \equiv \sum_{1\leqslant|a|\leqslant m_B-n_A} L^A_{B,a}(x)D^a u^B = b^A(x) \tag{2.1}$$

(1)See for instance [1].

where L^A_B a linear operator of order[2] $m_B - n_A$, summation over B and a is made and we denote as usual:

$$a = \alpha_1, \ldots, \alpha_n, \qquad D^a = D^{|a|}_{\alpha_1 \cdots \alpha_n}, \qquad |a| = \alpha_1 + \cdots + \alpha_n.$$

We denote by H the principal part of L, represented in coordinates by the matrix of the terms of order $m_B - n_A$ in L^A_B (such a term may be absent):

$$H^A_B(x, D)u^B \equiv \sum_{|a|=m_B-n_A} H^A_{B,a}(x) D^a u^B.$$

GKL call wave any solution of the homogeneous system ($b \equiv 0$) associated with L.

2.2. Asymptotic waves. — Let $u^{(r)}(x,\xi)$, $r = 0, 1, \ldots$ be a family of smooth fields defined on $V \times \mathbb{R}$. Let ω be a real parameter (called frequency by analogy with the WKB expansions). Let φ be a real function on V called phase. GKL consider a formal series on $V \times \mathbb{R}$ of the form

$$u^B(x,\xi) = \sum_{r=0}^{\infty} \omega^{-m_B - r} u^{B,r}(x,\xi). \tag{2.2}$$

For any field v on $V \times \mathbb{R}$ it holds that:

$$D_\alpha \{v(x,\xi)\}_{\xi=\omega\varphi(x)} = \{D_\alpha v(x,\xi) + \omega \varphi_\alpha v'(x,\xi)\}_{\xi=\omega\varphi(x)}$$

with

$$v' \equiv \frac{\partial v}{\partial \xi}, \qquad \varphi_\alpha \equiv \frac{\partial \varphi}{\partial x^\alpha}.$$

Inserting this identity in the formal computation of the action of the linear operator L on the formal series $u^B(x,\xi)_{\xi=\omega\varphi(x)}$ gives a formal series in powers of ω. The first term reads (summation in a and B, but not in A which labels the equation):

$$\sum_{|a|=m_B-n_A} \omega^{-n_A} H^A_{B,a}(x) \varphi^a \Big[\Big(\frac{\partial}{\partial \xi}\Big)^{m_B - n_A} u^{B,0}(x,\xi)\Big]_{\xi=\omega\varphi(x)}. \tag{2.3}$$

Definition 1. — A GKL asymptotic wave is a formal series of the type (2.2) such that the formal series obtained by its insertion in (2.1) is identically zero.

Neglecting terms irrelevant in the treatment obtained by n_A integrations with respect to ξ of each equation, the annulation of the term (2.3) is deduced from the equation

$$\sum_{|a|=m_B-n_A} H^A_{B,a}(x)\varphi^a \widetilde{u}^{B,0}(x,\xi) = 0, \qquad \widetilde{u}^B \equiv \Big(\frac{\partial}{\partial \xi}\Big)^{m_B} u^{B,0}.$$

[2] It can be shown that any linear system can be written under this form without modifying its characteristic polynomial. The numbers m and n are called Leray - Volevic indices.

A necessary and sufficient condition for these equations to have a solution $\widetilde{u}^{(0)}(x,\xi) \not\equiv 0$ is the vanishing of the following determinant:

$$\Delta(\varphi) \equiv \operatorname{Det}\Big(\sum_{|a|=m_B-n_A} H^A_{B,a}(x)\varphi^a\Big) = 0, \tag{2.4}$$

i.e. that $D\varphi$ be a solution of the characteristic (eikonal) equation of the operator L.

The phase φ being so chosen the first term $u^{(0)}$ of the asymptotic wave must be such that $\widetilde{u}^{(0)}$ belongs to the kernel of the linear homogeneous system:

$$\sum_{|a|=m_B-n_A} H^A_{B,a}(x)\varphi^a \widetilde{u}^{B,0}(x,\xi) = 0. \tag{2.5}$$

hence, supposing that the dimension of this kernel is 1 (simple characteristic), $u^{B,0}$ must be of the form

$$\widetilde{u}^{B,0} = U(x,\xi)h^B(x)$$

with h a particular solution of the system (2.5), depending only on x, and U a scalar function on $V\times\mathbb{R}$.

GKL show then that U must satisfy a linear propagation equation along the rays of the phase φ by writing the next term in the expansion, coefficient of ω^{-n_A-1}. Indeed the vanishing of this term reads (after n_A integrations with respect to ξ, $\widehat{a}_i$ means that α_i has been suppressed from the sequence a)

$$\begin{aligned}\sum_{|a|=m_B-n_A} \Big\{H^A_{B,a}(x)\varphi^a\widetilde{u}^{B,1}(x,\xi) + H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}\Big(\frac{\partial}{\partial\xi}\Big)^{m_B-1} D_{\alpha_i}u^{B,0}(x,\xi)\Big\}\\ + \sum_{|a|=m_B-n_A-1} L^{1,A}_{B,a}\varphi^a\Big(\frac{\partial}{\partial\xi}\Big)^{m_B-1} u^{B,0}(x,\xi) = 0.\end{aligned} \tag{2.6}$$

Since the determinant (2.4) is zero this equation can have a solution $\widetilde{u}^{(1)}$ only if the right hand side is orthogonal to the kernel $h^T(x)$ of the transposed linear system. Replacing $(\partial/\partial\xi)^{m_B-1}u^{B,0}$ by $\widehat{U}(x,\xi)h(x)$, with $\widehat{U}$ a primitive of U with respect to ξ leads to an ordinary first order differential system for $\widehat{U}$:

$$h^T_A(x)\{H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}D_{\alpha_i}[(\widehat{U}(x,\xi)h^B(x)] + \sum_{|a|=m_B-n_A-1} L^{1,A}_{B,a}\varphi^a\widehat{U}(x,\xi)h^B(x)\} = 0. \tag{2.7}$$

The identity

$$h^T_A(x)H^A_{B,\widehat{a}_i}(x)\varphi^{\widehat{a}_i}D_{\alpha_i}h^B(x) \equiv D_{\alpha_i}\Delta(\varphi)$$

shows that the system is a propagation system for $\widehat{U}$ along the rays of the phase φ, bicharacteristics of the operator L.

When U is determined, solution of (2.7), the second term $u^{(1)}$ is determined, up to a solution $U^{(1)}(x,\xi)h(x)$, by solving the linear equation (2.6), and integration with respect to ξ.

GKL show that an analogous procedure can be applied to annul the following terms in the expansion, and a formal asymptotic series can be constructed, through always linear systems and integration. Such an asymptotic series give approximate solutions to any order in ω, under smoothness assumptions of the coefficients.

2.3. Quasilinear systems. — The GKL construction has been extended to quasilinear first order systems in [**2**] by using a Taylor expansion of the coefficients in a neighbourhood of a solution (background). The equation for U contains then derivatives along the rays of the background and derivatives with respect to ξ. It leads to "dispersions of signals" if the system does not satisfy the Boillat - Lax exceptionnality condition. Due to the non linearity it is in general possible to obtain asymptotic approximate solutions of the given system only by truncating the series at first order in ω.

In the next sections we will consider quasilinear second order systems, with characteristic determinant possibly identically zero, and apply the results to some physical fields.

3. Quasilinear second order systems

3.1. Definitions. — We consider quasilinear second order systems with unknown a set of tensor fields u on a C^∞ manifold V. We do not write an explicit dependence in x, though it may exist. The system reads:

$$F(u, Du, D^2u) \equiv G(u, Du) \cdot D^2u + f(u, Du) = 0. \tag{3.1}$$

where D is the covariant derivative in some given pseudo riemannian smooth metric on V.

In index notations, with $u \equiv (u^A), A = 1, \dots, N$, and x^α local coordinates on V the system reads:

$$F^A(u, Du, D^2u) \equiv G_B^{A,\alpha\beta}(u, Du) D^2_{\alpha\beta} u^B + f^A(u, Du)) = 0.$$

The system is said to be quasi diagonal if

$$G_B^{A,\alpha\beta}(u, Du) \equiv g^{\alpha\beta}(u, Du)\delta_B^A$$

with δ_B^A the Kronecker delta. The fundamental field equations (Yang Mills, Einstein) are not quasidiagonal if a particular gauge is not chosen.

3.2. Asymptotic solutions

3.2.1. Definitions. — A high frequency wave on V is a tensor field of the type

$$u(x) = \underline{u}(x) + \omega^{-1}\{v(x, \xi)\}_{\xi = \omega\varphi(x)} \tag{3.2}$$

with $\underline{u}$ a tensor field on V, called background, v a tensor field of the same type as $\underline{u}$, but depending on a real parameter $\xi \in \mathbb{R}$, ω a real parameter ("frequency"), and φ a real function (phase).