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OPTIMAL RESULTS FOR THE TWO DIMENSIONAL NAVIER-STOKES EQUATIONS WITH LOWER REGULARITY ON THE DATA

by

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Abstract. — We establish existence and uniqueness of solutions in the anisotropic Sobolev space $H^{1,1/2}$ to the two dimensional Navier-Stokes equations with data in $H^{-1,-1/2}$. Our results give a new elementary proof for and extend some of recent results of G. Grubb.

***Résumé* (Résultats optimaux pour les équations de Navier-Stokes en dimension 2 avec des données initiales peu régulières)**

On établit l'existence et l'unicité des solutions dans l'espace de Sobolev anisotrope $H^{1,1/2}$ pour les équations de Navier-Stokes en dimension 2 avec des données dans $H^{-1,-1/2}$. Nos résultats donnent une preuve élémentaire nouvelle de résultats récents de G. Grubb, tout en les complétant.

1. Introduction

Working with divergence free vectorfields, the Navier-Stokes equations take the form

$$u_t - \Delta_x u + (u \cdot \nabla)u = f. \tag{1}$$

In two space dimensions it is known, since the pioneering works by J. Leray [**LE**], O.A. Ladyzhenskaya [**L1**], [**L2**],[**L3**] and J.L. Lions and G. Prodi [**LP**], that under suitable boundary conditions, (1) has a unique solution $u \in L^2(\mathbb{R}, H^1)$ for $f \in L^2(\mathbb{R}, H^{-1})$.

Later on these results have been complemented in various ways. In a recent interesting paper [**G**] G. Grubb gives general existence and uniqueness theorems for the Navier-Stokes equations in scales of L^p-Sobolev, Bessel potential and Besov spaces,

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using fairly complicated pseudo differential operator techniques. In two space dimensions with zero initial data her results include that under suitable conditions, given a source term $f \in H^{-1,-1/2}$ there exists a unique solution $u \in H^{1,1/2}$. However, in order to obtain existence it is assumed in [**G**] that the data is small enough in norm.

In the present note we give an elementary proof of the existence and uniqueness in the case of $f \in H^{-1,-1/2}$ and $u \in H^{1,1/2}$, without assuming smallness on the data. Our approach is in the spirit of the seminal work by J. Leray which appeared in *Acta Mathematica* in 1934, and it is based on the method of [**F1**], which turns out to be adaptable also to this situation. We refer to Theorem 1 below for the precise statement of our result. The main improvement in our result, compared to previously known results, is the regularity gain of the extra half derivative in time for the solution, and at the same time the corresponding wider range of possible irregularities for the source term.

Additional motivation for reconsidering the case $f \in H^{-1,-1/2}$ is provided by the fact that the corresponding result is optimal in a certain sense. Namely, the solution and the source spaces are in complete duality and, moreover, the difference in the smoothness corresponds exactly to the order of the non-linear operator in the respective variables.

An advantage of our approach is that it is completely elementary and self contained. Moreover, it appears to be possible to generalize it to certain situations, where other methods probably fail. For example our argument goes through unchanged if we replace the Laplacian in (1) with a uniformly elliptic linear operator having measurable coefficients (see the remark at the end of the paper).

We briefly mention some recent related results on the two-dimensional case. The papers of H. Amann [**A**] and [**A1**] apply interpolation arguments and semigroup methods to consider data with strong irregularity in space, but nonnegative smoothness in time. The paper [**MS1**] of J. Mattingly and Y. Sinai uses direct estimates on Fourier series to reprove and extend previous results in the case of very high regularity in space. As we are concerned with low regularity for the data in time, it is of interest here to observe that Brickmont, Kupiainen and Lefevre [**BKL1**] use the method of [**MS1**] to treat a very specific situation, where the smoothness of the source term corresponds to that of the white noise process, which barely fails to be locally $H^{-1/2}$ in time (see also [**BKL2**], [**MS2**] and [**KS**] in this connection). We refer to [**G**], [**MS1**], [**FT**], [**A**], [**A1**], [**L3**] and [**T**] and their references for further results.

The structure of the proof (and the note) is as follows: In the first section we consider the linearized operator and prove that it yields an isomorphism between the right spaces. To this end we apply simple Fourier analysis in connection with the Hilbert transform and the half-derivative operator; the conclusion is obtained by an application of the Lax-Milgram lemma. In the second section we first verify a suitable a priori estimate for the solution, which is based on a simple non-homogeneous Sobolev imbedding theorem (see Lemma 3). The existence of a solution in the non-linear case is now deduced from a simple finite dimensional approximation combined with an application of the theory of the Brouwer mapping degree. In turn, the proof of the uniqueness follows the classical lines.

2. The linear case

In the linear case there is no restriction on the space dimension. Let Ω be an open, bounded and connected set in $\mathbb{R}^n$, let $Q = \Omega \times \mathbb{R}$ and let

$$H^{1,1/2}(Q,\mathbb{R}^n) := \Big\{ u \in L^2(Q,\mathbb{R}^n);\, \frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial u}{\partial x_i} \in L^2(Q,\mathbb{R}^n) \text{ for } 1 \leqslant i \leqslant n \Big\}. \tag{2}$$

Here the half-derivative $\partial_+^{1/2} u/\partial t^{1/2}$ corresponds to the Fourier-multiplier $(i\tau)^{1/2}$, where τ is the Fourier frequency of t and we use the principal branch of the square root. In a similar manner, the half-derivative $\partial_-^{1/2} u/\partial t^{1/2}$ corresponds to the multiplier $(-i\tau)^{1/2}$. We obtain a Hilbert space with the norm

$$\Big(\iint_Q |u|^2 + \Big|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\Big|^2 + \sum_{i=1}^n \Big|\frac{\partial u}{\partial x_i}\Big|^2 dxdt \Big)^{1/2}.$$

By the Poincaré inequality, for elements in the closure of compactly supported functions, this is equivalent to the norm

$$\|u\|_{H^{1,1/2}} = \Big(\iint_Q \Big|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\Big|^2 + \sum_{i=1}^n \Big|\frac{\partial u}{\partial x_i}\Big|^2 dxdt \Big)^{1/2}, \tag{3}$$

which we will use henceforth. Let $\mathcal{V}(Q,\mathbb{R}^n)$ denote the space of divergence free (in the space variables) $\mathcal{D}(Q,\mathbb{R}^n)$-testfunctions. Here $\mathcal{D}$ stands for infinitely differentiable and compactly supported test functions.

We denote the closure in the $H^{1,1/2}(Q,\mathbb{R}^n)$ norm of $\mathcal{V}(Q,\mathbb{R}^n)$ by

$$V_0^{1,1/2}(Q,\mathbb{R}^n) := \overline{\mathcal{V}(Q,\mathbb{R}^n)}. \tag{4}$$

The restriction of an element ξ in the dual space $V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ to the space of divergence free testfunctions $\mathcal{V}(Q,\mathbb{R}^n)$ can be extended to a (non-unique) distribution in $H_0^{1,1/2}(Q,\mathbb{R}^n)^*$.

Lemma 1. — *Given $\xi \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$, there exist functions $f_0, f_1, \ldots, f_n \in L^2(Q,\mathbb{R}^n)$ such that*

$$\langle \xi, \Phi \rangle = \Big\langle \frac{\partial_+^{1/2} f_0}{\partial t^{1/2}} + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, \Phi \Big\rangle; \quad \Phi \in \mathcal{V}(Q,\mathbb{R}^n). \tag{5}$$

Furthermore, given $\varepsilon > 0$ we may always arrange so that $\|f_0\|_{L^2(Q,\mathbb{R}^n)} \leqslant \varepsilon$ (the extension might then of course have a bigger norm).

Proof. — The statement of the Lemma is a direct consequence of the Hahn-Banach theorem, which yields the stated expression apart from the control on the L^2-norm of f_0. For that end take a smooth test function g so that $\|f_0-g\|_{L^2(Q,\mathbb{R}^n)} \leqslant \varepsilon$. Replace f_0 by f_0-g and f_1 by $f_1+\displaystyle\int_{-\infty}^{x_1} \frac{\partial_+^{1/2} g(s,x_2,\ldots,x_n,t)}{\partial t^{1/2}}ds$ in the stated expression (where g is continued as zero outside of Q). This proves the Lemma since one easily verifies that the last term belongs to $L^2(Q,\mathbb{R}^n)$. □

Let $T_0 : V_0^{1,1/2}(Q,\mathbb{R}^n) \to V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ be the operator

$$T_0(u) = \frac{\partial u}{\partial t} - \Delta u, \tag{6}$$

defined by

$$\langle T_0 u, \Phi\rangle = \iint_Q \Big[\Big(\frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial_-^{1/2}\Phi}{\partial t^{1/2}}\Big) + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}, \frac{\partial \Phi}{\partial x_i}\Big)\Big] dxdt\,; \quad \Phi \in V_0^{1,1/2}(Q,\mathbb{R}^n). \tag{7}$$

One should here observe that $(\partial_-^{1/2})^* = \partial_+^{1/2}$ and $(\partial_+^{1/2})^2 = \partial$, and thus (7) is obtained from (6) by a formal integration by parts. The operator T_0 is a well-defined continuous linear operator since the above expression defines a continuous bilinear form on $V_0^{1,1/2}(Q,\mathbb{R}^n)$, as is seen by using the observation that $\partial_-^{1/2} = h\partial_+^{1/2}$, where h is the Hilbert transform. Recall that the Hilbert transform corresponds to the unimodular Fourier multiplier $-i\,\mathrm{sgn}(\tau)$, and hence h is an isometry on L^2.

Definition 1. — We say that a subspace of $L^2(Q,\mathbb{R}^n)$ is *invariant* if it is invariant under the Hilbert transform h in the time direction.

An invariant subspace will then be invariant also under the action of the operator H, defined by $H(u) = 1/\sqrt{2}(u - h(u))$. Observe that the paper [**F1**] introduced the operator H^α, which for the choice $\alpha = 1/4$ corresponds to our H.

The following simple result in the linear case forms the cornerstone of our later treatment of the fully nonlinear equation. It corresponds to the simplest linear case considered in [**F1**, Section 4.1], whence we leave for the reader some easy computational elements of the argument. Recall that an operator $T : V \to V^*$ is coercive if there exists a constant $C > 0$ such that $\langle Tu, u\rangle \geqslant C\|u\|_V^2$ for all $u \in V$.

Proposition 1. — *Let V be a closed invariant subspace of $V_0^{1,1/2}(Q,\mathbb{R}^n)$ and let $f \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$. Then there exists a unique $u_V(f) \in V$ such that*

$$\langle T_0(u_V), \Phi\rangle = \langle f, \Phi\rangle \quad \textit{for all } \Phi \in V.$$

Proof. — The operator $H : V \to V$ is obviously an isometry as it corresponds to a unimodular Fourier multiplier in the time direction. In particular, it maps divergence free distributions to divergence free distributions.

Recall next certain additional basic properties of the Hilbert transform h and the half derivatives. First of all, h is an isometry on L^2 with the property $h \circ h = -\mathrm{Id}$.

Moreover, one has that $\int_{-\infty}^{\infty}(u,h(u))dt = 0$ assuming that $u \in L^2$. One also has that $\int_{-\infty}^{\infty}(\partial_+^{1/2}u, \partial_-^{1/2}u)dt = 0$ and $\int_{-\infty}^{\infty}(\partial_+^{1/2}u, \partial_-^{1/2}h(u))dt = -\int_{-\infty}^{\infty}|\partial_+^{1/2}u|^2 dt$ assuming that the integrals are well-defined. The latter equality is a consequence of the identity $\partial_-^{1/2}h = -\partial_+^{1/2}$. Using the above facts, a straightforward computation shows that the operator $H^* \circ T_0$ (defined by the natural bilinear form on $V \times V$) is a coercive linear operator from V to V^*. By the Lax-Milgram lemma it is an isomorphism, whence the same is true for T_0.

Finally, for the readers comfort we clarify the role of the restrictions of operators in the above argument. Let us denote by P_V the orthogonal projection on V in $V_0^{1,1/2}(Q,\mathbb{R}^n)$, so that $P_{V^*} = P_V^*$ is the orthogonal projection on V^* (which is thus identified with a closed subspace of $(V_0^{1,1/2}(Q,\mathbb{R}^n))^*$. In precise terms the above proof yields that the operator $(P_{V^*}H^*T_0)_{|V} : V \to V^*$ is an isomorphism. However, since H in an isometry on the whole space and $H : V \to V$ is bijective, the same is true for $H^* : V^* \to V^*$. As we have $P_V H = HP_V$, it also holds that $P_{V^*}H^* = H^*P_{V^*}$. We may now deduce that $(P_{V^*}T_0)_{|V} : V \to V^*$ is an isomorphism, and this is equivalent to the statement of the Theorem. □

We explicitly state the special case

Corollary 1. — *The operator* $T_0 : V_0^{1,1/2}(Q,\mathbb{R}^n) \to V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ *is an isomorphism.*

Concerning best approximations we have

Proposition 2. — *Let* $V \subset W$ *be two closed invariant subspaces in* $V_0^{1,1/2}(Q,\mathbb{R}^n)$, *let* $f \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ *and let* $u_V(f) \in V$, $u_W(f) \in W$ *be the corresponding solutions from Proposition 1. Then*

$$\|u_V - u_W\|_{H^{1,1/2}} \leqslant 2\|\Phi - u_W\|_{H^{1,1/2}} \quad \textit{for all } \Phi \in V. \tag{8}$$

Proof. — From

$$\langle T_0(u_V - u_W), H(u_V - u_W + u_W - \Phi)\rangle = 0\,; \quad \Phi \in V, \tag{9}$$

one computes as in the proof of Proposition 1 to obtain the inequality

$$a^2 + b^2 \leqslant 2(ac + bd)$$

with

$$a^2 = \iint_Q \left|\frac{\partial_+^{1/2}(u_V - u_W)}{\partial t^{1/2}}\right|^2 dxdt, \qquad b^2 = \iint_Q \left|\frac{\partial(u_V - u_W)}{\partial x_i}\right|^2 dxdt,$$

$$c^2 = \iint_Q \left|\frac{\partial_+^{1/2}(u_W - \Phi)}{\partial t^{1/2}}\right|^2 dxdt, \qquad \text{and} \quad d^2 = \iint_Q \left|\frac{\partial(u_W - \Phi)}{\partial x_i}\right|^2 dxdt,$$

The result is now a consequence of the Cauchy-Schwarz inequality. □

Lemma 2. — *There exists a sequence* $V_1 \subset V_2 \subset V_3 \subset \ldots$ *of finite dimensional (closed) invariant subspaces of* $V_0^{1,1/2}(Q,\mathbb{R}^n)$ *such that* $\overline{(\cup_{i=1}^{\infty} V_i) \cap \mathcal{D}(Q,\mathbb{R}^n)} = V_0^{1,1/2}(Q,\mathbb{R}^n)$.