Séminaires & Congrès
9, 2004, p. 173–183

SHEAVES: FROM LERAY TO GROTHENDIECK AND SATO

by

Pierre Schapira

Abstract. — We show how the ideas of Leray (sheaf theory), Grothendieck (derived categories) and Sato (microlocal analysis) lead to the microlocal theory of sheaves which allows one to reduce many problems of linear partial differential equations to problems of microlocal geometry. Moreover, sheaves on Grothendieck topologies are a natural tool to treat growth conditions which appear in Analysis.

***Résumé* (Faisceaux: de Leray à Grothendieck et Sato).** — Nous montrons comment les idées de Leray (théorie des faisceaux) Grothendieck (catégories dérivées) et Sato (analyse microlocale) conduisent à la théorie microlocale des faisceaux qui permet de réduire de nombreux problèmes d'équations aux dérivées partielles linéaires à des problèmes de géométrie microlocale. Les faisceaux sur les topologies de Grothendieck sont de plus un outil naturel pour traiter les conditions de croissance qui apparaissent en Analyse.

1. Introduction

The "Scientific work" of Jean Leray has recently been published [**7**]. It is divided in three volumes:

(a) Topologie et théorème du point fixe (algebraic topology),

(b) Équations aux dérivées partielles réelles et mécanique des fluides (non linear analysis),

(c) Fonctions de plusieurs variables complexes et équations aux dérivées partielles holomorphes (linear analytic partial differential equations, LPDE for short).

As we shall see, (a) and (c) are in fact closely related, and even complementary, when translated into the language of sheaves with a dose of homological algebra. Recall that sheaf theory, as well as the essential tool of homological algebra known under the vocable of "spectral sequences", were introduced in the 40's by Leray. I do

2000 *Mathematics Subject Classification*. — 35A27, 32C38.
***Key words and phrases*.** — Sheaves, D-modules, microsupport, microlocal analysis.

not intend to give an exhaustive survey of Leray's fundamental contributions in these areas of Mathematics. I merely want to illustrate by some examples the fact that his ideas, combined with those of Grothendieck [**1**] and Sato [**10**], [**11**], lead to an algebraic and geometric vision of linear analysis, what Sato calls "Algebraic Analysis".

I will explain how the classical "functional spaces" treated by the analysts in the 60's are now replaced by "functorial spaces", that is, sheaves of generalized holomorphic functions on a complex manifold X or, more precisely, complexes of sheaves $R\mathcal{H}om\,(G, \mathcal{O}_X)$, where G is an $\mathbb{R}$-constructible sheaf on the real underlying manifold to X, the seminal example being that of Sato's hyperfunctions [**10**]. I will also explain how a general system of LPDE is now interpreted as a coherent $\mathcal{D}_X$-module $\mathcal{M}$, where $\mathcal{D}_X$ denotes the sheaf of rings of holomorphic differential operators [**3**], [**11**].

The study of LPDE with values in a sheaf of generalized holomorphic functions is then reduced to that of the complex $R\mathcal{H}om\,(G, F)$, where $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is the complex of holomorphic solutions of the system $\mathcal{M}$.

At this stage, one can forget that one is working on a complex manifold X and dealing with LPDE, keeping only in mind two geometrical informations, the micro-support of G and that of F (see [**4**]), this last one being nothing but the characteristic variety of $\mathcal{M}$.

However, classical sheaf theory does not allow one to treat usual spaces of analysis, much of which involving growth conditions which are not of local nature, and to conclude, I will briefly explain how the use of Grothendieck topologies, in a very special and easy situation, allows one to overcome this difficulty. References are made to [**4**] and [**5**].

2. The Cauchy-Kowalevsky theorem, revisited

At the heart of LPDE is the Cauchy-Kowalevsky theorem (C-K theorem, for short). Let us recall its classical formulation, and its improvement, by Schauder, Petrowsky and finally Leray. As we shall see later, the C-K theorem, in its precise form given by Leray, is the only analytical tool to treat LPDE. All other ingredients are of topological or algebraic nature, sheaf theory and homological algebra.

The classical C-K theorem is as follows. Consider an open subset X of $\mathbb{C}^n$, with holomorphic coordinates $(z_1, \dots, z_n)$, and let Y denote the complex hypersurface with equation $\{z_1 = 0\}$. Let P be a holomorphic differential operator of order m. Hence

$$P = \sum_{|\alpha| \leqslant m} a_\alpha(z)\partial_z^\alpha$$

where $\alpha = (\alpha_1 \dots \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the $a_\alpha(z)$'s are holomorphic functions on X, and ∂_z^α is a monomial in the derivations $\partial/\partial z_i$.

One says that Y is non-characteristic if $a_{(m,0\dots,0)}$, the coefficient of $\partial_{z_1}^m$, does not vanish.

The Cauchy problem is formulated as follows. Given a holomorphic function g on X and m holomorphic functions $h = (h_0, \dots, h_{m-1})$ on Y, one looks for f holomorphic in a neighborhood of Y in X, solution of

$$\begin{cases} Pf = g, \\ \gamma_Y(f) = (h), \end{cases}$$

where $\gamma_Y(f) = (f|_Y, \partial_1 f|_Y, \dots, {\partial_1}^{m-1} f|_Y)$ is the restriction to Y of f and its $(m-1)$ first derivative with respect to z_1.

The C-K theorem asserts that if Y is non-characteristic with respect to P, the Cauchy problem admits a unique solution in a neighborhood of Y. Schauder and Petrovsky realized that the domain of existence of f depends only on X and the principal symbol of P, and Leray gave a precised version of this theorem:

***Theorem 2.1* (The C-K theorem revisited by Leray)**. — *Assume that X is relatively compact in $\mathbb{C}^n$ and the coefficients a_α are holomorphic in a neighborhood of $\overline{X}$. Assume moreover that $a_{m,0\dots,0} \equiv 1$. Then there exists $\delta > 0$ such that if g is holomorphic in a ball $B(a, R)$ centered at $a \in Y$ and of radius R, with $B(a, R) \subset X$, and (h) is holomorphic in $B(a, R) \cap Y$, then f is holomorphic in the ball $B(a, \delta R)$ of radius δR.*

This result seems purely technical, and its interest is not obvious. However it plays a fundamental role in the study of propagation, as illustrated by Zerner's result below.

To state it, we need to work free of coordinates. The principal symbol of P, denoted by $\sigma(P)$, is defined by

$$\sigma(P)(z; \zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha.$$

This is indeed a well-defined function on T^*X, the complex cotangent bundle to X. Identifying X to $X_\mathbb{R}$, the real underlying manifold, there is a natural identification of $(T^*X)_\mathbb{R}$ and the real cotangent bundle $T^*(X_\mathbb{R})$. The condition that Y is non-characteristic for P may be translated by saying that $\sigma(P)$ does not vanish on the conormal bundle to Y outside the zero-section, and one defines similarly the notion of being non characteristic for a real hypersurface.

***Proposition 2.2* ([13])**. — *Let Ω be an open set in X with smooth boundary S (hence S is a real hypersurface of class C^1 and Ω is locally on one side of S). Assume that S is non-characteristic with respect to P. Let f be holomorphic in Ω and assume that Pf extends holomorphically through the boundary S. Then f extends itself holomorphically through the boundary S.*

The proof is very simple (see also [**2**]). Using the classical C-K theorem, we may assume that $Pf = 0$. Then one solves the homogeneous Cauchy problem $Pf = 0$, $\gamma_Y(f) = \gamma_Y(f)$, along complex hyperplanes closed to the boundary. The precised C-K theorem tells us that the solution (which is nothing but f by the uniqueness) is holomorphic in a domain which "makes an angle", hence crosses S for Y closed enough to S.

A similar argument shows that it is possible to solve the equation $Pf = g$ is the space of functions holomorphic in Ω in a neighborhood of each $x \in \partial\Omega$, and with some more work one proves

Theorem 2.3. — *Assume that $\partial\Omega$ is non-characteristic with respect to P. Then for each $k \in \mathbb{N}$, P induces an isomorphism on $H^k_{X\setminus\Omega}(\mathcal{O}_X)|_{\partial\Omega}$.*

3. Microsupport

The conclusion of Theorem 2.3 may be formulated in a much more general framework, forgetting both PDE and complex analysis.

Let X denote a *real* manifold of class C^∞, let k be a field, and let F be a bounded complex of sheaves of k-vector spaces on X (more precisely, F is an object of $D^b(k_X)$, the bounded derived category of sheaves on X). As usual, T^*X denotes the cotangent bundle to X.

Definition 3.1. — The microsupport $SS(F)$ of F is the closed conic subset of T^*X defined as follows. Let U be an open subset of T^*X. Then $U \cap SS(F) = \varnothing$ if and only if for any $x \in X$ and any real C^∞-function $\varphi : X \to \mathbb{R}$ such that $\varphi(x) = 0, d\varphi(x) \in U$, one has:

$$(R\Gamma_{\varphi\geqslant 0}(F))_x = 0.$$

In other words, F has no cohomology supported by the closed half spaces whose conormals do not belong to its microsupport.

Let X be a complex manifold, P a holomorphic differential operator and let *Sol*(P) be the complex of holomorphic solutions of P:

$$\mathit{Sol}(P) := 0 \to \mathcal{O}_X \underset{P}{\longrightarrow} \mathcal{O}_X \to 0,$$

then Theorem 2.3 reads as:

$$SS(\mathit{Sol}(P)) \subset \mathit{char}(P). \tag{3.1}$$

This result can easily been extended to general systems (determined or not) of LPDE.

Let $\mathcal{D}_X$ denote the sheaf of rings of holomorphic differential operators, and let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module. Locally on X, $\mathcal{M}$ may be represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right. By classical arguments of analytic geometry (Hilbert's syzygies theorem), one shows that $\mathcal{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathcal{M}^\bullet := 0 \to \mathcal{D}_X^{N_r} \to \cdots \to \mathcal{D}_X^{N_1} \underset{\cdot P_0}{\longrightarrow} \mathcal{D}_X^{N_0} \to 0.$$

The complex of holomorphic solutions of $\mathcal{M}$, denoted $Sol(\mathcal{M})$, (or better in the language of derived categories, $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$), is obtained by applying $\mathcal{H}om_{\mathcal{D}_X}(\cdot, \mathcal{O}_X)$ to $\mathcal{M}^\bullet$. Hence

$$Sol(\mathcal{M}) := 0 \to \mathcal{O}_X^{N_0} \xrightarrow[P_0\cdot]{} \mathcal{O}_X^{N_1} \to \cdots \mathcal{O}_X^{N_r} \to 0,$$

where now $P_0\cdot$ operates on the left.

One defines naturally the characteristic variety of $\mathcal{M}$, denoted $char(\mathcal{M})$, a closed complex analytic conic subset of T^*X. For example, if $\mathcal{M}$ has a single generator u with relation $\mathcal{I}u = 0$, where $\mathcal{I}$ is a locally finitely generated ideal of $\mathcal{D}_X$, then

$$char(\mathcal{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0 \ \ \forall P \in \mathcal{I}\}.$$

Using purely algebraic arguments, one deduces from (3.1):

Theorem 3.2. — $SS(Sol(\mathcal{M})) \subset char(\mathcal{M})$.

In fact, one can also prove that the inclusion above is an equality.

4. Functorial spaces

In the sixties, people used to work in various spaces of generalized functions on a real manifold. The situation drastically changed with Sato's definition of hyperfunctions by a purely cohomological way. Recall that on a real analytic manifold M of dimension n, the sheaf $\mathcal{B}_M$ is defined by

$$\mathcal{B}_M = H^n_M(\mathcal{O}_X) \otimes or_M$$

where X is a complexification of M and or_M denotes the orientation sheaf on M. Let $\mathbb{C}_{XM}$ denote the constant sheaf on M with stalk $\mathbb{C}$ extended by 0 on $X \smallsetminus M$. By Poincaré's duality,

$$R\mathcal{H}om\,(\mathbb{C}_{XM}, \mathbb{C}_X) \simeq or_{M/X}[n]$$

where $or_{M/X} \simeq or_M$ is the (relative) orientation sheaf and $[n]$ means a shift in the derived category of sheaves. An equivalent definition of hyperfunctions is thus given by

$$\mathcal{B}_M = R\mathcal{H}om\,(D'_X\mathbb{C}_{XM}, \mathcal{O}_X) \tag{4.1}$$

where $D'_X = R\mathcal{H}om\,(\cdot, \mathbb{C}_X)$ is the duality functor.

The importance of Sato's definition is twofold: first, it is purely algebraic (starting with the analytic object $\mathcal{O}_X$), and second it highlights the link between real and complex geometry.

Let $\mathcal{A}_M$ denote the sheaf of real analytic functions on M, that is, $\mathcal{A}_M = \mathbb{C}_{XM} \otimes \mathcal{O}_X$. We have the isomorphism

$$\mathcal{A}_M \simeq R\mathcal{H}om\,(D'_X\mathbb{C}_{XM}, \mathbb{C}_X) \otimes \mathcal{O}_X,$$