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DUAL ELLIPTIC PLANES

by

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Abstract. — An elliptic plane is a complex projective plane V equipped with an elliptic structure E in the sense of Gromov (generalization of an almost complex structure), which is tamed by the standard symplectic form. The space V^* of surfaces of degree 1 tangent to E (E-lines) is again a complex projective plane. We define on V^* a structure of elliptic plane E^*, such that to each E-curve one can associate its dual in V^*, which is an E^*-curve. Also, the bidual (V^{**}, E^{**}) is naturally isomorphic to (V, E).

***Résumé* (Plans elliptiques duaux).** — Un plan elliptique est un plan projectif complexe équipé d'une structure elliptique E au sens de Gromov (généralisation d'une structure quasi-complexe), qui est positive par rapport à la forme symplectique standard. L'espace V^* des surfaces de degré un tangentes à E (E-droites) est de nouveau un plan projectif complexe. Nous définissons sur V^* une structure de plan elliptique E^*, telle qu'à toute E-courbe on peut associer sa duale dans V^*, qui est une E^*-courbe. En outre, le bidual (V^{**}, E^{**}) est naturellement isomorphe à (V, E).

Introduction

Let V be a smooth oriented 4-manifold, which is a rational homology $\mathbb{CP}^2$ (*i.e.* $b_2(V) = 1$), and let J be an almost complex structure on V which is homologically equivalent to the standard structure J_0 on $\mathbb{CP}^2$. This means that there is an isomorphism $H^*(V) \to H^*(\mathbb{CP}^2)$ (rational coefficients) which is positive on H^4 and sends the Chern class $c_1(J)$ to $c_1(J_0)$.

By definition, a J-*line* is a J-holomorphic curve (or J-*curve*) of degree 1. By the *positivity of intersections* [**McD2**], it is an embedded sphere. We denote by V^* the set of J-lines.

Now assume that J is *tame*, *i.e.* positive with respect to some symplectic form ω, and also that V^* is nonempty. Then M. Gromov [**G**, 2.4.A] (*cf.* also [**McD1**]) has

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proved that by two distinct points $x, y \in V$ there passes a unique J-line $L_{x,y} \in V^*$, depending smoothly on (x, y); also, for any given $P \in \mathrm{Gr}_1^J(TV)$, the Grassmannian of J-complex lines in TV, there exists a unique J-line $L_P \in V^*$ tangent to P. Furthermore, V is oriented diffeomorphic to $\mathbb{CP}^2$, ω is isomorphic to $\lambda\omega_0$ for some positive λ so that J is homotopic to J_0. Finally, V^* has a natural structure of compact oriented 4-manifold; although it is not explicitly stated in [**G**], the above properties of V^* imply that it is also oriented diffeomorphic to $\mathbb{CP}^2$.

***Remark* (J. Duval)**. — The dependence of L_P upon P is continuous but not smooth. However, when p is fixed, the map $P \in \mathrm{Gr}_1^J(T_pV) \approx \mathbb{CP}^1 \mapsto L_P$ has quasiconformal components in any smooth chart of V^* given by intersections with two J-lines. For more details, see [**D**, p. 4-5].

Later, Taubes [**T1, T2**] proved that the hypothesis that V^* be nonempty is unnecessary, so that all the above results hold when J is tame. We shall call (V, J) an *almost complex projective plane*.

Following [**G**, 2.4.E], these facts can be extended to the case of an *elliptic structure* on V, *i.e.* one replaces $\mathrm{Gr}_1^J(TV)$ by a suitable submanifold E of the Grassmannian of oriented 2-planes $\widetilde{Gr}_2(TV)$. Such a structure is associated to a *twisted almost complex structure* J, which is a fibered map from TV to itself satisfying $J_v^2 = -\mathrm{Id}$ but such that J_v is not necessarily linear.

An elliptic structure on V gives rise to a notion of *E-curve*, *i.e.* a surface $S \subset V$ (not necessarily embedded or immersed) whose tangent plane at every point is an element of E (for the precise definitions, see section 2). It will be called *tame* if there exists a symplectic form ω strictly positive on each $P \in E$.

In Gromov's words, "all facts on J-curves extend to E-curves with an obvious change of terminology". In particular, let V be a rational homology $\mathbb{CP}^2$ equipped with a tame elliptic structure E so that (V, E) is homologically equivalent to $(\mathbb{CP}^2, \mathrm{Gr}_1^{\mathbb{C}}(T\mathbb{CP}^2))$. Then one can define the space V^* of E-lines (E-curves of degree 1), and prove that all the above properties still hold (see section 3). In particular, V and V^* are oriented diffeomorphic to $\mathbb{CP}^2$.

We shall call (V, E) with the above properties an *elliptic projective plane*. If $C \subset V$ is an E-curve, we define its *dual* $C^* \subset V^*$ by $C^* = \{L_{T_vC} \mid v \in C\}$. A more precise definition is given in section 4; one must require that no component of C be contained in an E-line. The main new result of this paper is then the following.

Theorem. — *Let (V, E) be an elliptic projective plane. Then there exists a unique elliptic structure $E^* \subset \widetilde{\mathrm{Gr}}_2(TV^*)$ on V^* with the following property: if $C \subset V$ is an E-curve which has no component contained in an E-line, then its dual $C^* \subset V^*$ is an E^*-curve.*

Furthermore, (V^*, E^*) is again an elliptic projective plane. Finally, the bidual (V^{**}, E^{**}) can be canonically identified with (V, E), and $C^{**} = C$ for every E-curve C.

If E comes from an almost complex structure, one may wonder if this is also the case for E^*, equivalently if the associated twisted almost complex structure J^* is linear on each fiber. Ben McKay has proved that this happens only if J is integrable, *i.e.* isomorphic to the standard complex structure on $\mathbb{CP}^2$: see the end of the Introduction.

The theorem above enables us to extend to J-curves in $\mathbb{CP}^2$ (for a tame J) some classical results obtained from the theory of dual algebraic curves. For instance, one immediately obtains the Plücker formulas, which restrict the possible sets of singularities of J-curves.

Such results could be interesting for the symplectic isotopy problem for surfaces in $\mathbb{CP}^2$ [**Sik2, Sh**]. And maybe also for the topology of a symplectic 4-manifold X, in view of the result of D. Auroux [**Aur**] showing that X is a branched covering of $\mathbb{CP}^2$, provided one could rule out negative cusps in the branch locus.

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This idea has also been discovered independently by Ben McKay, who made a very deep study of elliptic structures (which may exist in any even dimension for V) from the point of view of exterior differential systems (see the references at the end and also his web site). He uses the terminology "generalized Cauchy-Riemann equations" and "generalized pseudoholomorphic curves".

In particular, he proved that the submanifold E giving the structure is equipped with a canonical almost complex structure. He also gave a positive answer to a conjecture that I had made (see Section 5): if the elliptic structures on V and on its dual V^* are both almost complex, then they are integrable (and thus V is isomorphic to $\mathbb{CP}^2$ with the standard complex structure).

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Structure of the paper. — In section 1, we study elliptic surfaces in the Grassmannian of oriented 2-planes of a 4-dimensional real vector space. In section 2 we study elliptic structures on a 4-manifold, *i.e.* fibrations in elliptic surfaces in the tangent spaces. In section 3 we define and study elliptic projective planes. Most of the statements and all the ideas in these three sections are already in Gromov's paper (see especially [**G**, 2.4.E and 2.4.A]), except what regards singularities, where we give more precise results in the vein of [**McD2**] and [**MW**].

In section 4 we prove the main result.

In section 5 we give a special case of a more general general result of McKay: a tame almost complex structure J on $V = \mathbb{CP}^2$ such that the elliptic structure on V^* is not almost complex (and thus V^* has no natural almost complex structure).

Finally in section 6 we prove the Plücker formulas for E-curves and in particular for J-curves.

1. Elliptic surfaces in a Grassmannian

1.A. Definition. Associated complex lines. — Let T be an oriented real vector space of dimension 4. We denote by $G(T) = \widetilde{\mathrm{Gr}}_2(T)$ the Grassmannian of oriented 2-planes. Recall that for each $P \in G(T)$, the tangent plane $T_P G(T)$ is canonically identified with $\mathrm{Hom}(P, T/P)$.

By definition, an *elliptic surface* in $G(T)$ is a smooth, closed, connected and embedded surface X such that for every $P \in X$ one has

$$T_P X \smallsetminus \{0\} \subset \mathrm{Isom}_+(P, T/P).$$

Lemma. — *Let P_1, P_2, P_3 be three oriented real planes ($\mathbb{R}$-vector spaces of dimension 2), and*

$$\phi : P_1 \longrightarrow \mathrm{Hom}(P_2, P_3)$$

be a linear map such that $\phi(P_1 \smallsetminus \{0\}) \subset \mathrm{Isom}_+(P_2, P_3)$. Then there exists unique complex structures j_1, j_2, j_3 on P_1, P_2, P_3, making them complex lines, compatible with the orientations, and such that the restriction $\phi : P_1 \to \mathrm{im}(\phi)$ is a complex isomorphism onto $\mathrm{Isom}_{\mathbb{C}}(P_2, P_3)$, i.e.

$$(*) \qquad \phi(p_1) \circ j_2 = j_3 \circ \phi(p_1), \quad \phi(j_1 p_1) = \phi(p_1) \circ j_2 = j_3 \circ \phi(p_1).$$

Proof of the Lemma. — We prove the uniqueness first. Let j_1, j_2, j_3 have the desired properties. Let (p_1^1, p_1^2) be an oriented base of P_1, and let

$$u = \phi(p_1^1)^{-1} \phi(p_1^2) \in \mathrm{GL}_+(P_2).$$

The hypothesis implies that u has eigenvalues $a \pm ib$ with $b > 0$. Replacing p_1^2 by $(p_1^2 - a p_1^1)/b$, we can obtain that these eigenvalues are $\pm i$.

Note that u belongs to the plane $P = \phi(p_1^1)^{-1}[\mathrm{im}(\phi)] \subset \mathrm{End}(P_2)$. This plane is generated by Id and $j_2 = \phi(p_1^1)^{-1} \phi(j_1 p_1^1)$, thus the fact that u has eigenvalues $\pm i$ implies $j_2 = \varepsilon u$ with $\varepsilon = \pm 1$.

Thus $j_1 p_1^1 = \varepsilon p_1^2$, and since $(p_1^1, j_1 p_1^1)$ and (p_1^1, p_1^2) are both oriented bases of P_1, we have $\varepsilon = 1$, thus

$$\begin{aligned} &j_2 = \phi(p_1^1)^{-1} \phi(p_1^2), \\ &j_1(p_1^1) = p_1^2, \quad j_1(p_1^2) = -p_1^1, \\ &j_3 = \phi(p_1^2) \circ \phi(p_1^1)^{-1}. \end{aligned}$$

This proves the uniqueness.

Conversely, it is easy to see that these formulas define complex structures compatible with the orientations, and that $(*)$ is satisfied. $\square$

Applying this lemma, we obtain complex structures on $T_P X$, P, T/P, making them complex lines. We shall denote by

– $j_{X,P}$ the structure on $T_P X$,
– j_P and $j_P^{\perp}$ the structures on P and T/P.

By the integrability of almost complex structures on surfaces, X inherits a well-defined structure of Riemann surface.

1.B. Elliptic surfaces and complex structures. — The first example of elliptic surface is a Grassmannian $\mathrm{Gr}_1^J(T)$ of complex J-lines for a positive complex structure J on T.

We now prove that every elliptic surface is deformable to such a $\mathrm{Gr}_1^J(T)$. More precisely, denote by $\mathcal{J}(T)$ the space of positive complex structures, and $\mathcal{E}(T)$ the space of elliptic surfaces. Then the embedding $\mathcal{J}(T) \to \mathcal{E}(T)$ just defined admits a retraction by deformation. In particular, X is always diffeomorphic to $\mathbb{CP}^1$ and thus biholomorphic to $\mathbb{CP}^1$.

To prove this, we fix a Euclidean metric on V and replace $\mathcal{J}(T)$ by the subspace $\mathcal{J}_0(T)$ of isometric structures, to which it retracts by deformation. The space of 2-vectors $\Lambda^2 T$ has a decomposition $\Lambda^2 T = \Lambda^2_+ T \oplus \Lambda^2_- T$ into self-dual and antiself-dual vectors. The Grassmannian $G(T)$ is identified with $S^2_+ \times S^2_- \subset \Lambda^2_+ T \times \Lambda^2_- T$ by sending a plane P to $(\sqrt{2}(x\wedge y)_+, \sqrt{2}(x \wedge y)_-)$ where (x,y) is any positive orthonormal basis. We denote by $P = \phi(u_+, u_-)$ the plane associated to (u_+,u_-). Identifying T/P with $P^{\perp}$, the canonical isomorphism

$$T_{u_+} S^2_+ \times T_{u_-} S^2_- \longrightarrow \mathrm{Hom}(P, P^{\perp})$$

sends (α_+, α_-) to A such that

$$A.\xi = *(\xi \wedge (\alpha_+ + \alpha_-)).$$

This can be seen by working in a unitary oriented basis of T, (e_1, e_2, e_3, e_4) such that $u_\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$. This leads to unitary oriented bases of $T_{u_+} S^2_+$ and $T_{u_-} S^2_-$:

$$v^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4), \quad w^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

Still working in these bases, one gets

$$\det A = -\|\alpha_+\|^2 + \|\alpha_-\|^2,$$

(beware the signs!). Thus an elliptic structure is given by a surface $X \subset S^2_+ \times S^2_-$ such that the projections $p_\pm : X \to S^2_\pm$ satisfy

– dp_- is an isomorphism at all points of X,
– $\|dp_+ \circ (dp_-)^{-1}\| < 1$ at all points of X.