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UNIFORMLY QUASICONFORMAL PARTIALLY HYPERBOLIC SYSTEMS

BY Clark BUTLER AND Disheng XU

ABSTRACT. – We study smooth volume-preserving perturbations of the time-1 map of the geodesic flow ψ_t of a closed Riemannian manifold of dimension at least three with constant negative curvature. We show that such a perturbation has equal extremal Lyapunov exponents with respect to volume within both the stable and unstable bundles if and only if it embeds as the time-1 map of a smooth volume-preserving flow that is smoothly orbit equivalent to ψ_t. Our techniques apply more generally to give an essentially complete classification of smooth, volume-preserving partially hyperbolic diffeomorphisms which satisfy a uniform quasiconformality condition on their stable and unstable bundles and have either compact center foliation with trivial holonomy or are obtained as perturbations of the time-1 map of an Anosov flow.

RÉSUMÉ. – Nous étudions les perturbations lisses préservant le volume de l'application temps-un du flot géodésique ψ_t d'une variété riemannienne fermée de dimension au moins égale à trois et de courbure négative constante. Nous montrons que pour une telle perturbation, les exposants de Lyapunov extrémaux relativement au volume coïncident à la fois dans les sous-espaces stables et instables si et seulement si cette perturbation se plonge comme temps-un d'un flot lisse préservant le volume et dont les orbites sont conjuguées de manière lisse à celles de ψ_t. Nos techniques s'appliquent plus généralement pour donner une classification essentiellement complète des difféomorphismes lisses, partiellement hyperboliques préservant le volume et vérifient une condition de quasi-conformalité uniforme le long de leurs fibrés stables et instables qui, soit possèdent un feuilletage central compact avec une holonomie triviale, soit sont obtenus comme perturbations de l'application temps-un d'un flot d'Anosov.

1. Introduction

A surprising number of rigidity problems originally posed in negatively curved geometry turn out to have solutions that are dynamical in nature. We review one such story here: Sullivan proposed, following work of Gromov [24] and Tukia [41], that closed Riemannian manifolds of constant negative curvature and dimension at least 3 should be characterized up to isometry by the property that the geodesic flow acts *uniformly quasiconformally* on

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the unstable foliation [40]. Informally, the uniform quasiconformality property states that the flow does not distort the shape of metric balls inside of a given horosphere over a long period of time. Sullivan's conjecture was partially confirmed by the work of Kanai [30] who showed that among contact Anosov flows the geodesic flows of constant negative curvature manifolds are characterized up to C^1 orbit equivalence by a uniform quasiconformality. Later the minimal entropy rigidity theorem of Besson, Courtois, and Gallot [4] completed the proof of Sullivan's conjecture among many other outstanding conjectures in negatively curved geometry.

From a geometric perspective this completes the story, but from a dynamical perspective this raises many new questions. We see already in the work of Kanai that the dynamical version of this rigidity result holds for a larger class of Anosov flows than just geodesic flows. Sadovskaya initiated a program to extend these results further to smooth volume-preserving Anosov flows and diffeomorphisms [36], which was completed in a series of works by Fang ([18], [19], [20]) who obtained the following remarkable result: all smooth volume-preserving Anosov flows which are uniformly quasiconformal on the stable and unstable foliation are smoothly orbit equivalent either to the suspension of a hyperbolic toral automorphism or to the geodesic flow on the unit tangent bundle of a constant negative curvature closed Riemannian manifold. Thus we see that not even the contact structure of the flow is necessary to obtain dynamical rigidity for uniformly quasiconformal Anosov flows.

In a different direction one can ask whether the uniform quasiconformality condition can be relaxed to a condition that is more natural from the perspective of ergodic theory. This direction was pursued by the first author, who showed that for geodesic flows of $\frac{1}{4}$-pinched negatively curved manifolds, uniform quasiconformality can be derived from the significantly weaker dynamical condition of equality of all Lyapunov exponents with respect to volume on the unstable bundle [13].

Our principal goal is to show that for all of the rigidity phenomena derived from uniform quasiconformality above, *not even the structure of an Anosov flow is necessary*. Let us be more precise: consider a closed Riemannian manifold X of constant negative curvature with $\dim X \geq 3$. Let T^1X be the unit tangent bundle of X and let $\psi_t : T^1X \to T^1X$ denote the time-t map of the geodesic flow. This flow preserves a smooth volume m on T^1X known as the Liouville measure. Consider *any* smooth diffeomorphism f which is C^1-close to the time-1 map ψ_1 and which preserves the volume m. By the work of Hirsch, Pugh, and Shub [26], f is *partially hyperbolic*, meaning that there is a Df-invariant splitting $T(T^1X) = E^u \oplus E^c \oplus E^s$ where E^u is exponentially expanded by Df, E^s is exponentially contracted by Df, and the behavior of Df on the 1-dimensional center direction E^c (which is close to the flow direction for ψ_t) is dominated by the expansion and contraction on E^u and E^s respectively. We give a more precise definition in Section 2. We then choose a continuous norm $\|\cdot\|$ on E^u and define the *extremal Lyapunov exponents* of f on E^u by

$$\lambda_+^u(f) = \inf_{n\geq 1} \frac{1}{n}\int_M \log \|Df^n|E^u\|\, dm,$$
$$\lambda_-^u(f) = \sup_{n\geq 1} \frac{1}{n}\int_M \log \|(Df^n|E^u)^{-1}\|^{-1}\, dm.$$

We define $\lambda_+^s(f)$ and $\lambda_-^s(f)$ similarly with E^s replacing E^u.

THEOREM 1. – *There is a C^2-open neighborhood $\mathcal{U}$ of ψ_1 in the space of C^∞ volume-preserving diffeomorphisms of T^1X such that if $f \in \mathcal{U}$ and both of the equalities $\lambda^u_+(f) = \lambda^u_-(f)$ and $\lambda^s_+(f) = \lambda^s_-(f)$ hold, then there is a C^∞ volume-preserving flow φ_t with $\varphi_1 = f$. Furthermore φ_t is smoothly orbit equivalent to ψ_t.*

This theorem improves on the techniques used in the previous rigidity theorems in several fundamental ways. We are able to deduce uniform quasiconformality of the action of Df on E^u and E^s from equality of the extremal Lyapunov exponents entirely outside of the geometric context considered in [13] by using new methods. We then use this uniform quasiconformality to completely reconstruct the smooth flow φ_t in which f embeds as the time-1 map. We emphasize that for a typical perturbation f of ψ_1 the foliation $\mathcal{W}^c$ tangent to E^c (which is our candidate for the flowlines of φ_t) is only a continuous foliation of T^1X with no transverse smoothness properties. This is one of the many reasons that strong rigidity results in the realm of partially hyperbolic diffeomorphisms are quite rare. Our inspiration was an impressive rigidity theorem of Avila, Viana and Wilkinson which overcame this obstacle to show that if we take X to be a negatively curved surface instead and f a C^1-small enough C^∞ volume-preserving perturbation of the time-1 map ψ_1 such that the center foliation of f is absolutely continuous, then f is also the time-1 map of a smooth volume-preserving flow [3]. Our result can be viewed in an appropriate sense as the higher dimensional analog of this theorem.

We now explain the organization of the paper. The techniques used in the proof of Theorem 1 have much more general applications which can also be applied to the study of C^∞ volume-preserving partially hyperbolic diffeomorphisms which satisfy a uniform quasiconformality condition on their stable and unstable bundles and either have uniformly compact center foliation with trivial holonomy or are obtained as a perturbation of the time-1 map of an Anosov flow. These results are stated in Theorems 2 and 4 and Corollary 3 of Section 2 after we introduce some necessary terminology. In Section 3 we show that under a Lyapunov stability type result on the action of a partially hyperbolic diffeomorphism f on its center foliation, uniform quasiconformality implies that the holonomy maps of the center stable and center unstable foliations of f are quasiconformal. We use this to show that the center foliation of f is absolutely continuous. In Section 4 we prove the linearity of center holonomy for f between local unstable leaves in a suitable chart under some stronger assumptions on f; moreover, for such f the center, center (un)stable foliations are all smooth, see Section 5. In Section 6 we finish the proofs of Theorems 2 and 4 and Corollary 3. In Section 7 we finish the proof of Theorem 1 by deducing uniform quasiconformality from the condition of equality of extremal Lyapunov exponents. The arguments in Section 7 do not rely on the results of Sections 3, 4, 5 and 6 and may be read independently of the rest of the paper.

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