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François BERTELOOT & Fabrizio BIANCHI & Christophe DUPONT

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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DYNAMICAL STABILITY AND LYAPUNOV EXPONENTS FOR HOLOMORPHIC ENDOMORPHISMS OF $\mathbb{P}^k$

BY François BERTELOOT, Fabrizio BIANCHI
AND Christophe DUPONT

Abstract. – We introduce a notion of stability for equilibrium measures in holomorphic families of endomorphisms of $\mathbb{P}^k$ and prove that it is equivalent to the stability of repelling cycles and equivalent to the existence of some measurable holomorphic motion of Julia sets which we call equilibrium lamination. We characterize the corresponding bifurcations by the strict subharmonicity of the sum of Lyapunov exponents or the instability of critical dynamics and analyze how repelling cycles may bifurcate. Our methods deeply exploit the properties of Lyapunov exponents and are based on ergodic and pluripotential theory.

Résumé. – Nous introduisons une notion de stabilité pour les mesures d'équilibre des familles holomorphes d'endomorphismes de $\mathbb{P}^k$ et démontrons qu'elle est équivalente à la stabilité des cycles répulsifs et équivalente à l'existence d'un mouvement holomorphe mesurable des ensembles de Julia, appelé lamination d'équilibre. Nous caractérisons les bifurcations correspondantes par la sous-harmonicité stricte de la somme des exposants de Lyapunov ou par l'instabilité de la dynamique critique, nous analysons aussi comment les cycles répulsifs peuvent bifurquer. Nos méthodes reposent sur les propriétés des exposants de Lyapunov, sur la théorie ergodique et sur la théorie du pluripotentiel.

1. Introduction

1.1. Main definitions and results

In the early 1980's, Mañé, Sad and Sullivan [31] and Lyubich [29, 30] independently obtained fundamental results on the stability of holomorphic families $(f_\lambda)_{\lambda\in M}$ of rational maps of the Riemann sphere $\mathbb{P}^1$. They proved that the parameter space M splits into an open and dense *stability locus* and its complement, the *bifurcation locus*. They also obtained precise information on the distribution of hyperbolic parameters which led to the so-called hyperbolic conjecture. This conjecture asserts that hyperbolic maps are dense in the space of

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rational maps. The works of Douady and Hubbard on the Mandelbrot set provide a deeper understanding of these questions for the quadratic polynomial family.

In this theory, the finiteness of the critical set and the Picard-Montel theorem play a crucial role. They allow to characterize the stability of a parameter $\lambda_0 \in M$ by the stability of the critical orbits of the map f_{λ_0}. Equivalently, λ_0 is in the bifurcation locus if, after an arbitrarily small perturbation, there exists a repelling cycle capturing a critical orbit. The one-dimensional setting also permits, by means of the λ-lemma, to build holomorphic motions of Julia sets which conjugate the dynamics on connected components of the stability locus. The bifurcation locus also coincides with the closure of the parameters $\lambda \in M$ for which f_λ admits an unpersistent neutral cycle.

This article deals with bifurcations within holomorphic families of endomorphisms of $\mathbb{P}^k$ for $k \geq 1$. Let M be a connected complex manifold of dimension m. A holomorphic family of endomorphisms of $\mathbb{P}^k$ can be seen as a holomorphic mapping

$$f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k, \quad (\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

where the algebraic degree d of f_λ is larger than or equal to 2 and does not depend on λ. For instance, M can be the space $\mathcal{H}_d(\mathbb{P}^k)$ of all degree d holomorphic endomorphisms of $\mathbb{P}^k$, which is a Zariski open subset in some $\mathbb{P}^N$.

Our main result is Theorem 1.1 below, it asserts that different natural notions of stability are equivalent and leads to a coherent notion of bifurcation for holomorphic families f in $\mathbb{P}^k$. Our arguments exploit some ergodic and pluripotential tools as those developed in the works of Bedford-Lyubich-Smillie, Fornaess-Sibony, Briend-Duval, Dinh-Sibony on holomorphic dynamics on $\mathbb{P}^k$ or $\mathbb{C}^k$ (see the survey [21] for precise references). Let us recall that, for each $\lambda \in M$, we have an ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$ where μ_λ is the equilibrium measure of f_λ and J_λ is the topological support of μ_λ called the *Julia set*. The measure μ_λ enjoys a potential interpretation

$$\mu_\lambda = \left(dd^c_z\, g(\lambda, z) + \omega_{FS}\right)^k,$$

where g is the Green function of f and ω_{FS} the Fubini-Study form on $\mathbb{P}^k$. The repelling cycles of f_λ equidistribute the measure μ_λ and hence are dense in J_λ. However, in higher dimension, some repelling cycles may belong to the complement of J_λ. We denote by $L(\lambda) := \int_{\mathbb{P}^k} \log|\mathrm{Jac} f|\, d\mu_\lambda$ the sum of the Lyapunov exponents of μ_λ. This is a plurisubharmonic function on M which satisfies $L(\lambda) \geq k\frac{\log d}{2}$. Let $[C_f]$ denote the current of integration on the critical set C_f of f taking into account the multiplicities of f.

Our main result is as follows. The definitions occurring in (A), (C), (D) and (F) are explained below.

THEOREM 1.1. – *Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms where M is a simply connected open subset of the space $\mathcal{H}_d(\mathbb{P}^k)$ of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. Then the following assertions are equivalent:*

(A) *the repelling J-cycles move holomorphically over M,*
(B) *the function L is pluriharmonic on M,*
(C) *f admits an equilibrium web,*
(D) *f admits an equilibrium lamination,*
(E) *any $\lambda_0 \in M$ admits a neighborhood U such that* $\liminf_n d^{-kn} |(f^n)_*[C_f]|_{U \times \mathbb{P}^k} = 0$,

(F) *there are no Misiurewicz parameters in* M.

When $k = 2$, *these equivalences remain true for every simply connected manifold* M. *If one of these conditions is satisfied,* f *admits a unique equilibrium web* $\mathcal{M}$ *and* $\mathcal{M}\left(\mathcal{L}_1 \Delta \mathcal{L}_2\right) = 0$ *for any pair of equilibrium laminations* $\mathcal{L}_1, \mathcal{L}_2$ *of* f.

Theorem 1.1 leads us to define the *bifurcation current* of a holomorphic family of endomorphisms of $\mathbb{P}^k$ as the closed positive current $dd^c_\lambda L$, and the *bifurcation locus* as the support of this current. The family is *stable* if its bifurcation locus is empty, stability is clearly a local notion. This is coherent with the one-dimensional definition of the bifurcation current, due to DeMarco [15]. We stress that Theorem 1.1 stays partially true for general families (see Theorem 1.6).

Let us now specify the definitions. A central notion is the set

$$\mathcal{O} := \left\{\gamma : M \to \mathbb{P}^k : \gamma \text{ is holomorphic and } \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M\right\}.$$

The graph $\{(\lambda, \gamma(\lambda)),\ \lambda \in M\}$ of any element $\gamma \in \mathcal{O}$ is denoted Γ_γ. We endow $\mathcal{O}$ with the topology of local uniform convergence and note that f induces a continuous self-map

$$\mathcal{F} : \mathcal{O} \to \mathcal{O} \text{ given by } \mathcal{F} \cdot \gamma(\lambda) := f_\lambda(\gamma(\lambda)).$$

Definition 1.2. – *For every* $\lambda \in M$, *a repelling* J*-cycle of* f_λ *is a repelling cycle which belongs to* J_λ. *We say that these cycles* move holomorphically over M *if, for every period* n, *there exists a finite subset* $\{\rho_{n,j},\ 1 \le j \le N_n\}$ *of* $\mathcal{O}$ *such that* $\{\rho_{n,j}(\lambda),\ 1 \le j \le N_n\}$ *is precisely the set of* n *periodic repelling* J*-cycles of* f_λ *for every* $\lambda \in M$.

The holomorphic motion of repelling J-cycles over M also means that for every repelling periodic point $z_0 \in J_{\lambda_0}$ of f_{λ_0} there exists $\gamma \in \mathcal{O}$ such that $\gamma(\lambda_0) = z_0$ and $\gamma(\lambda)$ is a periodic repelling point of f_λ for every $\lambda \in M$.

Our notions of equilibrium webs and laminations are as follows.

Definition 1.3. – *An* equilibrium web *is a probability measure* $\mathcal{M}$ *on* $\mathcal{O}$ *such that*

1. $\mathcal{M}$ *is* $\mathcal{F}$*-invariant and its support is a compact subset of* $\mathcal{O}$,
2. *for every* $\lambda \in M$ *the probability measure* $\mathcal{M}_\lambda := \int_{\mathcal{O}} \delta_{\gamma(\lambda)}\, d\mathcal{M}(\gamma)$ *is equal to* μ_λ.

This notion is related to Dinh's theory of woven currents and somehow means that the measures $(\mu_\lambda)_{\lambda \in M}$ are holomorphically glued together. In this article we shall also say that the $(\mu_\lambda)_{\lambda \in M}$ move holomorphically when such a web exists.

Definition 1.4. – *An* equilibrium lamination *is a relatively compact subset* $\mathcal{L}$ *of* $\mathcal{O}$ *such that*

1. $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ *for every distinct* $\gamma, \gamma' \in \mathcal{L}$,
2. $\mu_\lambda\{\gamma(\lambda), \gamma \in \mathcal{L}\} = 1$ *for every* $\lambda \in M$,
3. Γ_γ *does not meet the grand orbit of the critical set of* f *for every* $\gamma \in \mathcal{L}$,
4. *the map* $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *is* d^k *to* 1.