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SUBGROUP DYNAMICS AND C^* -SIMPLICITY OF GROUPS OF HOMEOMORPHISMS

BY ADRIEN LE BOUDEC AND NICOLÁS MATTE BON

ABSTRACT. – We study the uniformly recurrent subgroups of groups acting by homeomorphisms on a topological space. We prove a general result relating uniformly recurrent subgroups to rigid stabilizers of the action, and deduce a C^* -simplicity criterion based on the non-amenability of rigid stabilizers. As an application, we show that Thompson’s group V is C^* -simple, as well as groups of piecewise projective homeomorphisms of the real line. This provides examples of finitely presented C^* -simple groups without free subgroups. We prove that a branch group is either amenable or C^* -simple. We also prove the converse of a result of Haagerup and Olesen: if Thompson’s group F is non-amenable, then Thompson’s group T must be C^* -simple. Our results further provide sufficient conditions on a group of homeomorphisms under which uniformly recurrent subgroups can be completely classified. This applies to Thompson’s groups F , T and V , for which we also deduce rigidity results for their minimal actions on compact spaces.

RÉSUMÉ. – Nous étudions les sous-groupes uniformément récurrents de groupes agissant par homéomorphismes sur un espace topologique. Nous prouvons un résultat général reliant les sous-groupes uniformément récurrents aux stabilisateurs rigides de l’action, et en déduisons un critère de C^* -simplicité basé sur la non moyennabilité des stabilisateurs rigides. Comme application, nous prouvons que le groupe de Thompson V est C^* -simple, de même que certains groupes d’homéomorphismes projectifs par morceaux de la droite réelle. Cela fournit des exemples de groupes finiment présentés qui sont C^* -simples et sans sous-groupes libres. Nous prouvons qu’un groupe branché est soit moyennable, soit C^* -simple. Nous prouvons également la réciproque d’un résultat de Haagerup et Olesen: si le groupe de Thompson F n’est pas moyennable alors le groupe de Thompson T est C^* -simple. Nos résultats fournissent de plus des conditions suffisantes sur un groupe d’homéomorphismes sous lesquelles les sous-groupes uniformément récurrents sont complètement compris. Cela s’applique aux groupes de Thompson, pour lesquels nous déduisons également des résultats de rigidité sur leurs actions sur des espaces compacts.

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1. Introduction

Let G be a second countable locally compact group. The set $\text{Sub}(G)$ of all closed subgroups of G admits a natural topology, defined by Chabauty in [17]. This topology turns $\text{Sub}(G)$ into a compact metrizable space, on which G acts continuously by conjugation.

The study of G -invariant Borel probability measures on $\text{Sub}(G)$, named *invariant random subgroups* (IRS's) after [2], is a fast-developing topic [2, 1, 27, 3]. In this paper we are interested in their topological counterparts, called *uniformly recurrent subgroups* (URSs) [24]. A uniformly recurrent subgroup is a closed, minimal, G -invariant subset $\mathcal{H} \subset \text{Sub}(G)$. We will denote by $\text{URS}(G)$ the set of uniformly recurrent subgroups of G . Every normal subgroup $N \trianglelefteq G$ gives rise to a URS of G , namely the singleton $\{N\}$. More interesting examples arise from minimal actions on compact spaces: if G acts minimally on a compact space X , then the closure of all point stabilizers in $\text{Sub}(G)$ contains a unique URS, called the *stabilizer URS* of $G \curvearrowright X$ [24].

When G has only countably many subgroups (e.g., if G is polycyclic), every IRS of G is atomic and every URS of G is finite, as follows from a standard Baire argument. Leaving aside this specific situation, there are important families of groups for which a precise description of the space $\text{IRS}(G)$ has been obtained. Considerably less is known about URSs. For example if G is a lattice in a higher rank simple Lie group, the normal subgroup structure of G is described by Margulis' Normal Subgroups Theorem. While Stuck-Zimmer's Theorem [64] generalizes Margulis' NST to IRS's, it is an open question whether a similar result holds for URSs of higher rank lattices, even for the particular case of $\text{SL}(3, \mathbb{Z})$ [24, Problem 5.4].

1.1. Micro-supported actions

In this paper we study the space $\text{URS}(G)$ for countable groups G admitting a faithful action $G \curvearrowright X$ on a topological space X such that for every non-empty open set $U \subset X$, the *rigid stabilizer* G_U , i.e., the pointwise stabilizer of $X \setminus U$ in G , is non-trivial. Following [16], such an action will be called *micro-supported*. Note that this implies in particular that X has no isolated points.

The class of groups admitting a micro-supported action includes Thompson's groups $F < T < V$ and many of their generalizations, groups of piecewise projective homeomorphisms of the real line [49], piecewise prescribed tree automorphism groups [41], branch groups (viewed as groups of homeomorphisms of the boundary of the rooted tree) [4], and topological full groups acting minimally on the Cantor set. These groups have uncountably many subgroups, and many examples in this class have a rich subgroup structure.

Our first result shows that many algebraic or analytic properties of rigid stabilizers are inherited by the uniformly recurrent subgroups of G . In the following theorem and everywhere in the paper, a uniformly recurrent subgroup $\mathcal{H} \in \text{URS}(G)$ is said to have a group property if every $H \in \mathcal{H}$ has the corresponding property.

THEOREM 1.1 (see also Theorem 3.5). – *Let G be a countable group of homeomorphisms of a Hausdorff space X . Assume that for every non-empty open set $U \subset X$, the rigid stabilizer G_U is non-amenable (respectively contains free subgroups, is not elementary amenable, is*

not virtually solvable, is not locally finite). Then every non-trivial uniformly recurrent subgroup of G has the same property.

This result has applications to the study of C^* -simplicity; see Section 1.2.

We obtain stronger conclusions on the uniformly recurrent subgroups of G under additional assumptions on the action of G on X . Recall that when X is compact and $G \curvearrowright X$ is minimal, the closure of all point stabilizers in $\text{Sub}(G)$ contains a unique URS, called the *stabilizer URS* of $G \curvearrowright X$, and denoted $\mathcal{S}_G(X)$ [24] (see Section 2 for details). The following result provides sufficient conditions under which $\mathcal{S}_G(X)$ turns out to be the unique URS of G , apart from the points $\{1\}$ and $\{G\}$ (hereafter denoted 1 and G). We say that $G \curvearrowright X$ is an *extreme boundary action* if X is compact and the action is minimal and extremely proximal (see §2.1 for the definition of an extremely proximal action).

THEOREM 1.2. – *Let X be a compact Hausdorff space, and let G be a countable group of homeomorphisms of X . Assume that the following conditions are satisfied:*

- (i) $G \curvearrowright X$ is an extreme boundary action;
- (ii) there is a basis for the topology consisting of open sets $U \subset X$ such that the rigid stabilizer G_U admits no non-trivial finite or abelian quotients;
- (iii) the point stabilizers for the action $G \curvearrowright X$ are maximal subgroups of G .

Then the only uniformly recurrent subgroups of G are 1 , G and $\mathcal{S}_G(X)$.

We actually prove a more general result, see Corollary 3.12. Examples of groups to which this result applies are Thompson's groups T and V , as well as examples in the family of groups acting on trees $G(F, F')$ (see Section 1.3 and Section 1.5). In particular, this provides examples of finitely generated groups G (with uncountably many subgroups) for which the space $\text{URS}(G)$ is completely understood. In the case of Thompson's groups, we deduce from this lack of URSs rigidity results about their minimal actions on compact spaces (see §1.3).

1.2. Application to C^* -simplicity

A group G is said to be C^* -simple if its reduced C^* -algebra $C_{\text{red}}^*(G)$ is simple. This property naturally arises in the study of unitary representations: G is C^* -simple if and only if every unitary representation of G that is weakly contained in the left-regular representation λ_G is actually weakly equivalent to λ_G [33]. Since amenability of a group G is characterized by the fact that the trivial representation of G is weakly contained in λ_G , a non-trivial amenable group is never C^* -simple.

The first historical C^* -simplicity result was Powers' proof that the reduced C^* -algebra of the free group \mathbb{F}_2 is simple [60]. The methods employed by Powers have then been generalized in several different ways, and various classes of groups have been shown to be C^* -simple. We refer to [33, Proposition 11] (see also the references given there), and to Corollary 12 therein for a list of important examples of groups to which these methods have been applied.

Problems related to C^* -simplicity recently received new attention [38, 8, 61, 41, 39, 31]. A characterization of C^* -simplicity in terms of boundary actions was obtained by Kalantar and Kennedy: a countable group G is C^* -simple if and only if G acts topologically freely on its Furstenberg boundary; equivalently, G admits some topologically free boundary action [38] (we recall the terminology in §2.1). By developing a systematic approach based on