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FUNDAMENTAL GROUPS OF F-REGULAR SINGULARITIES VIA F-SIGNATURE

BY Javier CARVAJAL-ROJAS, Karl SCHWEDE AND Kevin TUCKER

ABSTRACT. – We prove that the local étale fundamental group of a strongly F-regular singularity is finite. These results are analogous to results of Xu and Greb-Kebekus-Peternell for KLT singularities in characteristic 0. Our result is effective, we show that the reciprocal of the F-signature of the singularity gives a bound on the size of this fundamental group. To prove these results we develop new transformation rules for the F-signature under finite étale-in-codimension-one extensions. We also obtain purity of the branch locus over rings with mild singularities (particularly if the F-signature is $> 1/2$).

RÉSUMÉ. – Nous montrons que le groupe fondamental local étale d'une singularité F-régulière est fini. Ce théorème représente l'analogue en caractéristique p des résultats obtenus par Xu et Greb-Kebekus-Peternell pour les singularités KLT. Nous montrons que le cardinal du groupe fundamental est majoré par l'inverse de la F-signature de la singularité. En particulier, notre résultat principal est effectif. Pour cela, nous établissons des nouvelles formules de transformation de la F-signature par rapport aux extensions étale en codimension un. Nous obtenons également un nouveau critère de pureté du lieu de branchement sur les anneauux à singularités faibles. Ceci s'applique en particulier aux anneaux dont la F-signature est supérieure à 1/2.

1. Introduction

In [21, Question 26] J. Kollár asked whether if $(0 \in X)$ is the germ of a KLT singularity, then $\pi_1(X \setminus \{0\})$ is finite. In [41] C. Xu showed that this holds for the étale local fundamental group, in other words, for the profinite completion of $\pi_1(X \setminus \{0\})$. Building on this result, [11] proved the finiteness of the étale fundamental groups of the regular locus of KLT singularities (see also [35]). Over the past few decades, we have learned that KLT singularities are closely related to strongly F-regular singularities in characteristic $p > 0$, see [16, 15]. Hence it is natural to ask whether their local étale fundamental groups are also finite. We show that

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this is indeed the case. In fact, we find an upper bound for the size of the fundamental group in terms of a well studied invariant for measuring singularities in characteristic $p > 0$, the F-signature $s(R)$.

THEOREM A (Theorem 5.1). – *Let $(R, \mathfrak{m}, k)$ be a normal F-finite and strongly F-regular strictly Henselian*[(1)] *local domain of prime characteristic $p > 0$, with dimension $d \geq 2$. Then the étale fundamental group of the punctured spectrum of R, i.e., $\pi_1 := \pi_1^{\text{ét}}(\operatorname{Spec}^\circ(R))$, is finite. Furthermore, the order of π_1 is at most $1/s(R)$ and is prime to p. The same also holds for $\pi_1^{\text{ét}}(\operatorname{Spec}(R) \setminus Z)$ where $Z \subseteq \operatorname{Spec} R$ has codimension ≥ 2.*

Observe that unlike the characteristic zero situation, our characteristic $p > 0$ result is effective. We give an explicit bound on the size of the π_1. It is also worth noting that we are working with the étale fundamental group, not the *tame* fundamental group. Indeed, for R strongly F-regular, any finite Galois étale in codimension 1 local extension $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ must be tame everywhere. This was already implicitly observed in [31] but we make it precise here. Indeed, we note that p cannot divide $[K(S) : K(R)]$ if the residue fields are equal (Corollary 2.11).

The technical tool where F-regularity is used in our proof is a transformation rule for F-signature under finite étale-in-codimension-1-morphisms. The F-signature was introduced implicitly in [32] and explicitly in [18]. Roughly speaking, it measures how many different ways $R \hookrightarrow F_*^e R$ splits as e goes to infinity. Explicitly, if R has perfect residue field and $F_*^e R = R^{\oplus a_e} \oplus M$ as an R-module, where M has no free R-summands, then $s(R) = \lim_{e \to \infty} \dfrac{a_e}{p^{e \dim R}}$. Here are three quick facts:

- The limit $s(R)$ exists [37].
- $s(R) > 0$ if and only if R is strongly F-regular [1].
- $s(R) \leq 1$.

Note that there have been a number of transformation rules for F-signature under finite maps in the past. However, they were generally only an inequality (that went the wrong way for our purposes), or assumed that S is flat over R (or made other assumptions about R and S). See for instance [18, 42, 17, 37].

THEOREM B (Theorem 3.1). – *Let $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ be a module-finite local extension of F-finite d-dimensional normal local domains in characteristic $p > 0$, with corresponding extension of fraction fields $K \subseteq L$. Suppose $R \subseteq S$ is étale in codimension 1, and that R is strongly F-regular. Then if one writes $S = R^{\oplus f} \oplus M$ as a decomposition of R-modules so that M has no nonzero free direct summands, then $f = [\ell : k] \geq 1$ and the following equality holds:*

$$s(S) = \frac{[L : K]}{[\ell : k]} \cdot s(R).$$

Below, before Theorem D, we discuss how to still get precise transformation rules of F-signature even when $R \subseteq S$ is not necessarily étale in codimension 1.

By applying Theorem B in the case $k = \ell$, we see that $s(S) = [L : K] \cdot s(R)$. Since $s(S) \leq 1$, we immediately see that $[L : K] \leq 1/s(R)$. In other words, the reciprocal

(1) This just means it is Henselian with separably closed residue field.

of the F-signature $s(R)$ gives an upper bound on the generic rank of a finite local étale in codimension 1 extension with the same residue field. Theorem A then follows. We also obtain characteristic $p > 0$ corollaries similar to some of those in [11].

Because our bound on the size of the étale fundamental group is effective, we immediately obtain a new result on purity of the branch locus.

THEOREM C (Corollary 3.3). – *Suppose $Y \to X$ is a finite dominant map of F-finite normal integral schemes. If $s(\mathcal{O}_{X,P}) > 1/2$ for all $P \in X$ then the branch locus of $Y \to X$ has no irreducible components of codimension ≥ 2, in other words it is a divisor.*

In [5], the notion of F-signature of pairs was introduced. In Theorem 4.4, we obtain an analogous result to Theorem B in the context of pairs. Indeed, if (R, Δ) is a strongly F-regular pair, then this can be interpreted as follows. The reciprocal of $s(R, \Delta)$ gives an upper bound on the generic rank of a finite local extension $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ such that $\pi^*\Delta - \mathrm{Ram} \geq 0$ (here Ram is the ramification divisor on Spec S and $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$ is the induced map). By taking cones, this immediately yields the following characteristic $p > 0$ analog of the second main result of [41]. Here note that globally F-regular varieties are an analog of log-Fano varieties in characteristic zero [30].

THEOREM D (Corollary 4.8). – *Suppose that (X, Δ) is a globally F-regular projective pair over an algebraically closed field of characteristic $p > 0$. There is a number n such that every finite separable cover $\pi : Y \to X$ with $\pi^*\Delta - \mathrm{Ram} \geq 0$ has generic rank $[K(Y) : K(X)] \leq n$.*

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2. Preliminaries

CONVENTION 2.1. – Throughout this paper, all rings will be assumed to be Noetherian. They will all be characteristic $p > 0$ unless otherwise stated and they will all be F-finite. All schemes will be assumed to be Noetherian and separated. If R is an integral domain, then $K(R)$ will denote the fraction field of R (likewise with $K(X)$ if X is an integral scheme). Given a finite separable map of normal integral schemes $f : Y \to X$, we use Ram to denote the ramification divisor on Y.