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THE DYNAMICAL MORDELL-LANG CONJECTURE FOR POLYNOMIAL ENDOMORPHISMS OF THE AFFINE PLANE

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***Abstract.* —** In this paper we prove the Dynamical Mordell-Lang Conjecture for polynomial endomorphisms of the affine plane over the algebraic numbers. More precisely, let f be an endomorphism of the affine plan over the algebraic numbers. Let x be a point in the affine plan and C be a curve. If the intersection of C and the orbits of x is infinite, then C is periodic.

***Résumé* (*La conjecture de Mordell-Lang dynamique pour les applications polynomiales du plan affine*)**

Nous prouvons dans cet article la Conjecture Dynamique de Mordell-Lang pour les endomorphismes polynomiaux du plan affine sur les nombres algébriques. Plus précisément, soit f un endomorphisme du plan affine sur les nombres algébriques. Soient x un point dans le plan affine et C une courbe. Si l'intersection de C et les orbites de x est infinie, alors C est périodique.

CONTENTS

INTRODUCTION

1. The dynamical Mordell Lang conjecture

This article is concerned with the so-called *dynamical Mordell-Lang conjecture* that was proposed by Ghioca and Tucker in [**14**].

Dynamical Mordell-Lang Conjecture (see [**14**]). — *Let X be a quasi-projective variety defined over $\mathbb{C}$, let $f : X \to X$ be an endomorphism, and V be any subvariety of X. For any point $p \in X(\mathbb{C})$ the set $\{n \in \mathbb{N} \mid f^n(p) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions* [(1)].

This conjecture is inspired by the Mordell-Lang conjecture on subvarieties of semiabelian varieties (now a theorem of Faltings [**7**] and Vojta [**24**]), which says that if V is a subvariety of a semiabelian variety G defined over $\mathbb{C}$ and Γ is a finitely generated subgroup of $G(\mathbb{C})$, then $V(\mathbb{C}) \bigcap \Gamma$ is a union of at most finitely many translates of subgroups of Γ.

Observe that the dynamical Mordell-Lang conjecture implies the classical Mordell-Lang conjecture in the case $\Gamma \simeq (\mathbb{Z}, +)$.

It is also motivated by the Skolem-Mahler-Lech Theorem [**21**] on linear recurrence sequences. More precisely, suppose $\{A_n\}_{n\geq 0}$ is any recurrence sequence satisfying

$$A_{n+l} = F(A_n, \ldots, A_{n+l-1})$$

for all $n \geq 0$, where $l \geq 1$ and $F(x_0, \ldots, x_l) = \sum_{i=0}^{l-1} a_i x_i$ is a linear form on $\mathbb{C}^l$. The Skolem-Mahler-Lech Theorem asserts that the set $\{n \geq 0 \mid A_n = 0\}$ is a union of at most finitely many arithmetic progressions.

This statement is equivalent to the dynamical Mordell-Lang conjecture for the linear map $f : (x_0, \ldots, x_{l-1}) \mapsto (x_1, \ldots, x_{l-1}, F(x_0, \ldots, x_l))$ and the hyperplane $V = \{x_0 = 0\}$.

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1. An arithmetic progression is a set of the form $\{an + b \mid n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$. In particular, when $a = 0$, it contains only one point

2. The main results and comparison to previous results

Our goal is to prove this conjecture for *any* polynomial endomorphism of $\mathbb{A}^2_{\overline{\mathbb{Q}}}$.

Theorem 1. — *Let $f : \mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\overline{\mathbb{Q}}}$ be a polynomial endomorphism defined over $\overline{\mathbb{Q}}$. Let C be an irreducible curve in $\mathbb{A}^2_{\overline{\mathbb{Q}}}$ and p be a closed point in $\mathbb{A}^2_{\overline{\mathbb{Q}}}$. Then the set $\{n \in \mathbb{N} \mid f^n(p) \in C\}$ is a finite union of arithmetic progressions.*

Pick any polynomial $F(x, y) \in \overline{\mathbb{Q}}[x, y]$. By applying this result to the map

$$f : \mathbb{A}^2_{\overline{\mathbb{Q}}} \longrightarrow \mathbb{A}^2_{\overline{\mathbb{Q}}}, \quad (x, y) \longmapsto \big(y, F(x, y)\big)$$

and $C = \{x = 0\}$, we obtain the following corollary about recurrence sequences.

Corollary 2. — *Let $\{A_n\}_{n\geq 0}$ be a sequence of algebraic numbers satisfying*

$$A_{n+2} = F(A_n, A_{n+1})$$

for all $n \geq 0$, where $F(x, y) \in \overline{\mathbb{Q}}[x, y]$. Then the set $\{n \geq 0 \mid A_n = 0\}$ is a finite union of arithmetic progressions.

A direct induction on the dimension also yields the following

Theorem 3. — *For any non-constant polynomials $F_1, \dots, F_m \in \overline{\mathbb{Q}}[T]$, let us consider on $\mathbb{A}^m_{\overline{\mathbb{Q}}}$ the endomorphism*

$$f := \big(F_1(x_1), \dots, F_m(x_m)\big).$$

For any irreducible curve $C \subset \mathbb{A}^m_{\overline{\mathbb{Q}}}$ defined over $\overline{\mathbb{Q}}$ and any point $p \in \mathbb{A}^m(\overline{\mathbb{Q}})$, the set

$$\big\{n \geq 0 \mid f^n(p) \in C\big\}$$

is a finite union of arithmetic progressions.

The dynamical Mordell-Lang conjecture has received quite a lot of attention in the recent years and our theorems are closely related to several known results.

Bell, Ghioca and Tucker [**1**] proved the Dynamical Mordell-Lang conjecture for étale maps of quasiprojective varieties of arbitrary dimension, thereby generalizing the Skolem-Mahler-Lech Theorem [**21**] on linear recurrence sequences. The core of their argument is to work in a p-adic field and to analyze the dynamics in a quasi-periodic region where they are able to construct suitable invariant curves. Afterwards, the author [**27**] proved the dynamical Mordell-Lang conjecture for birational endomorphisms of the affine plane. The techniques in [**27**] are of a very different flavour. Particularly, in [**27**], we obtained a new proof of the dynamical Mordell-Lang conjecture for polynomial automorphisms of $\mathbb{A}^2$ which are *not* conjugated to an automorphism of some projective surface. However, we relied on Bell, Ghioca and Tucker's result in some cases, especially in the case of affine automorphisms of $\mathbb{A}^2$. In this paper, we develop the techniques used in [**27**] in a more general situation and use them more systematically.