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TILTING MODULES AND THE p-CANONICAL BASIS

Simon Riche & Geordie Williamson

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TILTING MODULES AND THE p-CANONICAL BASIS

by Simon RICHE & Geordie WILLIAMSON

Abstract. — In this book we propose a new approach to tilting modules for reductive algebraic groups in positive characteristic. We conjecture that translation functors give an action of the (diagrammatic) Hecke category of the affine Weyl group on the principal block. Our conjecture implies character formulas for the simple and tilting modules in terms of the p-canonical basis, as well as a description of the principal block as the antispherical quotient of the Hecke category. We prove our conjecture for $\mathrm{GL}_n(\Bbbk)$ using the theory of 2-Kac-Moody actions. Finally, we prove that the diagrammatic Hecke category of a general crystallographic Coxeter group may be described in terms of parity complexes on the flag variety of the corresponding Kac-Moody group.

Résumé — Dans cet ouvrage nous proposons une nouvelle approche à l'étude des modules basculants pour les groupes algébriques réductifs sur des corps de caractéristique positive. Nous conjecturons que les foncteurs de translation induisent une action de la catégorie de Hecke (diagrammatique) du groupe de Weyl affine sur le bloc principal. Cette conjecture implique des formules de caractères pour les modules simples et les modules basculants en termes de la base p-canonique, ainsi qu'une description du bloc principal comme le quotient anti-sphérique de la catégorie de Hecke. Nous démontrons notre conjecture pour le groupe $\mathrm{GL}_n(\Bbbk)$ en utilisant la théorie des représentations des algèbres 2-Kac-Moody. Enfin, nous prouvons que la catégorie de Hecke diagrammatique d'un groupe de Coxeter cristallographique général peut être décrite en termes de faisceaux à parité sur la variété de drapeaux du groupe de Kac-Moody correspondant.

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CHAPTER 1

INTRODUCTION

1.1. Overview

In this book we give new conjectural character formulas for simple and indecomposable tilting modules for a connected reductive algebraic group in characteristic p, [1] and we prove our conjectures in the case of the group $\mathrm{GL}_n(\Bbbk)$ when $n \geq 3$. These conjectures are formulated in terms of the p-canonical basis of the corresponding affine Hecke algebra. They should be regarded as evidence for the philosophy that Kazhdan-Lusztig polynomials should be replaced by *p-Kazhdan-Lusztig polynomials* in modular representation theory. From this point of view several conjectures (Lusztig's conjecture, James' conjecture, Andersen's conjecture) become the question of agreement between canonical (or Kazhdan-Lusztig) and p-canonical bases.

In the general setting we prove that the new character formulas follow from a very natural conjecture of a more categorical nature, which has remarkable structural consequences for the representation theory of reductive algebraic groups. It is a classical observation that wall-crossing functors provide an action of the affine Weyl group W on the Grothendieck group of the principal block; in this way, the principal block gives a categorification of the antispherical module for W. We conjecture that this action can be categorified: namely, that the action of wall-crossing functors on the principal block gives rise to an action of the diagrammatic Bott-Samelson Hecke category attached to W as in [**31**]. From this conjecture we deduce the following properties.

1. The principal block is equivalent (as a module category over the diagrammatic Bott-Samelson Hecke category) to a categorification of the antispherical module defined by diagrammatic generators and relations.
2. The principal block admits a grading. Moreover, this graded category arises via extension of scalars from a category defined over the integers. Thus the principal block of any reductive algebraic group admits a "graded integral form".

1. This book is written under the assumption that p is (strictly) larger than the Coxeter number, and so p cannot be too small. However there is a variant of our conjectures for any p involving singular variants of the Hecke category. In particular, it seems likely that p-Kazhdan-Lusztig polynomials give correct character formulas for tilting modules in *any* characteristic (see Conjecture 1.4.3). We hope to return to this subject in a future work.

3. The (graded) characters of the simple and tilting modules are determined by the p-canonical basis in the antispherical module for the Hecke algebra of W.

From (1) one may describe the principal block in terms of parity sheaves on the affine flag variety, which raises the possibility of calculating simple and tilting characters topologically. Point (2) gives a strong form of "independence of p". Finally, point (3) implies Lusztig's character formula for large p.

We prove this "categorical" conjecture (hence in particular the character formulas) for the groups $\mathrm{GL}_n(\Bbbk)$ using the Khovanov-Lauda-Rouquier theory of 2-Kac-Moody algebra actions. We view this, together with the agreement with Lusztig's conjecture for large p and character formulas of Soergel and Lusztig in the context of quantum groups (see § 1.7 for details) as strong evidence for our conjecture.

1.2. The "categorical" conjecture

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic p with simply connected derived subgroup. We assume that $p > h$, where h is the Coxeter number of G. Let $T \subset B \subset G$ be a maximal torus and a Borel subgroup in G. Denote by $\mathbf{X} := X^*(T)$ the lattice of characters of T and by $\mathbf{X}^+ \subset \mathbf{X}$ the subset of dominant weights. Consider a regular block $\mathrm{Rep}_0(G)$ of the category of finite-dimensional algebraic G-modules, corresponding to a weight $\lambda_0 \in \mathbf{X}$ in the fundamental alcove, with its natural highest weight structure. If Φ is the root system of (G,T), $W_{\mathrm{f}} = N_G(T)/T$ is the corresponding Weyl group, and $W := W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$ is the affine Weyl group, then the simple, standard, costandard and indecomposable tilting objects in the highest weight category $\mathrm{Rep}_0(G)$ are all parametrized by $W \bullet \lambda_0 \cap \mathbf{X}^+$. (Here "$\bullet$" denotes "$p$-dilated dot action of W of $\mathbf{X}$," see § 3.1.) If ${}^{\mathrm{f}}W \subset W$ is the subset of elements w which are minimal in their coset $W_{\mathrm{f}}w$, then there is a natural bijection

$$ {}^{\mathrm{f}}W \xrightarrow{\sim} W \bullet \lambda_0 \cap \mathbf{X}^+ $$

sending w to $w \bullet \lambda_0$. In this way we can parametrize the simple, standard, costandard and indecomposable tilting objects in $\mathrm{Rep}_0(G)$ by ${}^{\mathrm{f}}W$, and denote them by $\mathbb{L}(x \bullet \lambda_0)$, $\Delta(x \bullet \lambda_0)$, $\nabla(x \bullet \lambda_0)$ and $\mathbb{T}(x \bullet \lambda_0)$ respectively. In particular, on the level of Grothendieck groups we have

$$ [\mathrm{Rep}_0(G)] = \bigoplus_{x \in {}^{\mathrm{f}}W} \mathbb{Z}[\nabla(x \bullet \lambda_0)]. \tag{1.2.1} $$

Let $S \subset W$ denote the simple reflections. To any $s \in S$ one can associate a "wall-crossing" functor Ξ_s by translating to and from an s-wall of the fundamental alcove. Consider the "antispherical" right $\mathbb{Z}[W]$-module $\mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[W_{\mathrm{f}}]} \mathbb{Z}[W]$, where $\mathbb{Z}_\varepsilon$ denotes the sign module for the finite Weyl group W_{f} (viewed as a right $\mathbb{Z}[W_{\mathrm{f}}]$-module). This module has a basis (as a $\mathbb{Z}$-module) consisting of the elements $1 \otimes w$ with $w \in {}^{\mathrm{f}}W$. Then we can reformulate (1.2.1) as an isomorphism

$$ \phi\colon \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[W_{\mathrm{f}}]} \mathbb{Z}[W] \xrightarrow{\sim} [\mathrm{Rep}_0(G)] \tag{1.2.2} $$