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L-groups and the Langlands program for covering groups: A historical introduction

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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS: A HISTORICAL INTRODUCTION

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Abstract. — In this joint introduction to the present *Astérisque* volume, we shall give a short discussion of the historical developments in the study of nonlinear covering groups, touching on their structure theory, representation theory and the theory of automorphic forms. This serves as a historical motivation and sets the scene for the papers in this volume. Our discussion is necessarily subjective and will undoubtedly leave out the contributions of many authors, to whom we apologize in earnest.

Résumé **(L-groupes et le programme de Langlands pour les revêtements de groupes réductifs : une introduction historique)**

Dans cette introduction au présent volume de la série *Astérisque*, nous allons donner une brève discussion historique de l'étude des revêtements non linéaires des groupes réductifs, concernant leur structure, la théorie de leurs représentations et la théorie de leurs formes automorphes. Cela constitue une motivation historique et définit le cadre pour les contributions de ce volume.

1. Generalities

A *locally compact group* will mean a locally compact, Hausdorff, second countable topological group. Let G be a locally compact group and A a locally compact abelian group. We are interested in central extensions of G by A. Let us first define this notion; our treatment in this section follows the classic paper of Moore [**83**].

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1.1. Definition. — A *central extension of* G *by* A is a short exact sequence:

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

such that

- E is a locally compact group;
- i is continuous and $i(A)$ is a closed subgroup of the center of E;
- p is continuous and induces a topological isomorphism $E/i(A) \cong G$.

Equivalently, the third condition above can be replaced by the requirement that p is continuous and open (cf. [**80**, p.96]). We will ultimately be interested in the case when A is finite.

1.2. Definition. — Let E_1 and E_2 be two extensions of G by A. An *equivalence* from E_1 to E_2 is a continuous homomorphism $\phi\colon E_1 \to E_2$ inducing the identity maps on A and G:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "="'] & E_1 \arrow[r, "p_1"] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, "="'] & 1 \\ 1 \arrow[r] & A \arrow[r] & E_2 \arrow[r, "p_2"] & G \arrow[r] & 1. \end{tikzcd}$$

By the open mapping theorem, an equivalence is necessarily a topological isomorphism.

Let the set of equivalence classes of central extensions of G by A be denoted by $\mathrm{CExt}(G, A)$. The set $\mathrm{CExt}(G, A)$ has a natural abelian group structure, as we now explain.

Given two extensions E_1 and E_2 of G by A, we set

$$E = \{(h_1, h_2) \in E_1 \times E_2 : p_1(h_1) = p_2(h_2)\}/\delta(A)$$

where $\delta(a) = (a, a^{-1})$ is the skew diagonal embedding. This is the quotient of the fiber product $E_1 \times_G E_2$ by the skew diagonal embedding. Then E is a central extension of G by A,

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1,$$

by defining $i(a) = (a, 1) = (1, a) \in E$ and $p(h_1, h_2) = p_1(h_1) = p_2(h_2)$.

This E is the so-called Baer sum of E_1 and E_2, written $E_1 \dotplus E_2$ and this operation makes $\mathrm{CExt}(G, A)$ into an abelian group. In other words, the equivalence class of E depends only on the equivalence classes of E_1 and E_2.

In the context of abstract groups, the abelian group $\mathrm{CExt}(G, A)$ was first studied by Schur (1904) who introduced the notion of Schur multipliers. In modern language, Schur had introduced the cohomology group $H^2(G, A)$. We will however not go so far back in time in our historical discussion; a modern survey of the central extensions of finite groups of Lie type can be found in [**88**].

1.3. Categorical point of view. — If we fix G and A as before, define $\mathsf{CExt}(G, A)$ to be the category whose objects are central extensions of G by A, and whose morphisms are equivalences. Since all equivalences are isomorphisms, the category $\mathsf{CExt}(G, A)$ is a groupoid. The Baer sum is functorial,

$$\dot{+}\colon \mathsf{CExt}(G, A) \times \mathsf{CExt}(G, A) \to \mathsf{CExt}(G, A),$$

making the category $\mathsf{CExt}(G, A)$ into a (strictly commutative) Picard category [**34**, Définition 1.4.2]

The neutral object in this category is the direct product $G \times A$. Given an object $E \in \mathsf{CExt}(G, A)$, a *splitting* of E is an equivalence (i.e., a morphism) from E to $G \times A$.

If $j\colon H \to G$ is a continuous homomorphism of locally compact groups, and $(E, i, p) \in \mathsf{CExt}(G, A)$, then we may *pull back* the extension E to define

$$j^*E = \{(h, e) \in H \times E : j(h) = p(e)\}.$$

Then $j^*E \in \mathsf{CExt}(H, A)$ by defining $i'\colon A \to j^*E$ by $i'(a) = (1, i(a))$ and $p'(h, e) = p(e)$.

If $f\colon A \to B$ is a continuous homomorphism of locally compact abelian groups, we may *push out* the extension (E, i, p) to define

$$f_*E = (B \times E)/\overline{\langle (f(a), i(a)^{-1}) : a \in A \rangle}.$$

Typically, f will be a closed map, and so it will not be necessary to take the closure in the quotient above. Then $f_*E \in \mathsf{CExt}(G, B)$ by defining $i''\colon B \to f_*E$ by $i''(b) = (b, 1)$ and $p''\colon f_*E \to G$ by $p''(b, e) = p(e)$.

Pullback and pushout define additive functors of Picard categories,

$$f_*\colon \mathsf{CExt}(G, A) \to \mathsf{CExt}(G, B), \quad f^*\colon \mathsf{CExt}(G, A) \to \mathsf{CExt}(H, A).$$

For isomorphism classes, these define homomorphisms of abelian groups,

$$f_*\colon \operatorname{CExt}(G, A) \to \operatorname{CExt}(G, B), \quad f^*\colon \operatorname{CExt}(G, A) \to \operatorname{CExt}(H, A).$$

1.4. Cohomological interpretation. — After the foundational work of Mackey [**73**], Moore wrote a series of papers [**82, 84, 85**] developing a cohomology theory for topological groups analogous to that for abstract groups. We summarize some of their results.

Moore defines for each $n \geq 0$ a cohomology group $H^n(G, A)$ using *measurable* cochains. These groups are functors which are covariant in A and contravariant in G. Note however that since the category of locally compact abelian groups is not an abelian category, this cohomology theory is not a derived functor cohomology theory. We describe the low degree cohomology groups concretely. Note that we are only interested in the case where A is trivial as a G-module. The 0-th cohomology group is $H^0(G, A) = A$. The first cohomology $H^1(G, A)$ is the group of continuous homomorphisms $G \to A$.

We describe $H^2(G, A)$ in more detail. Let $Z^2(G, A)$ be the group of measurable normalized 2-cocycles $z\colon G \times G \to A$; this means that $z(g, 1) = z(1, g) = 1$ for all

$g \in G$, and

$$z(g_1g_2, g_3)z(g_1, g_2) = z(g_1, g_2g_3)z(g_2, g_2) \text{ for all } g_1, g_2, g_3 \in G.$$

Let $C^1(G, A)$ be the group of normalized 1-cochains: measurable functions from G to A such that $f(1) = 1$. If $f \in C^1(G, A)$ is a measurable function, its coboundary $\partial f \in Z^2(G, A)$ is defined by

$$\partial f(g_1, g_2) = f(g_2) \cdot f(g_1g_2)^{-1} \cdot f(g_1)^{-1}.$$

The resulting cohomology group $H^2(G, A) = Z^2(G, A)/\partial C^1(G, A)$ is naturally isomorphic to $\mathrm{CExt}(G, A)$.

This can be understood categorically as follows. Consider the (small, strictly commutative Picard) category $\mathsf{H}^2(G, A)$, with objects set $Z^2(G, A)$, and where a morphism $z_1 \to z_2$ is defined to be an element $c \in C^1(G, A)$ such that $z_2 = z_1 + \partial c$. The Picard category structure arises from the abelian group structures on $Z^2(G, A)$ and $C^1(G, A)$. The isomorphism classes in $\mathsf{H}^2(G, A)$ form the cohomology group $H^2(G, A)$.

Describe a functor from $\mathsf{H}^2(G, A)$ to $\mathsf{CExt}(G, A)$ as follows: for an object $z \in Z^2(G, A)$, define an extension of G by A by $E = G \times A$, with multiplication

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1g_2, a_1a_2 \cdot z(g_1, g_2)),$$

and maps

$$i : a \mapsto (1, a) \in E \quad \text{and} \quad p : (g, a) \mapsto g \in G.$$

A theorem of Mackey [**73**, Théorème 2] gives E a natural topology such that the above defines a locally compact group, and an extension of G by A. If $c \colon z_1 \to z_2$ is a morphism in $\mathsf{H}^2(G, A)$, i.e., $z_2 = z_1 + \partial c$, then c defines an equivalence of central extensions $E_1 \to E_2$ by the formula $f(g, a) = (g, c(g) \cdot a)$. The work of Mackey and Moore implies that this gives an equivalence of Picard categories, which we like to call "incarnation":

$$\mathsf{Inc} \colon \mathsf{H}^2(G, A) \to \mathsf{CExt}(G, A).$$

A consequence is the isomorphism of abelian groups, $H^2(G, A) \cong \mathrm{CExt}(G, A)$.

Surjectivity of this isomorphism is obtained as follows. Given a central extension $A \hookrightarrow E \twoheadrightarrow G$, Mackey proves that one can find a measurable section $s \colon G \to E$ (i.e., so that $p \circ s = id$). This is the best one can hope for: one cannot find a continuous section in general. From s, one defines a measurable 2-cocycle by:

$$z(g_1, g_2) = s(g_1g_2) \cdot s(g_1)^{-1}s(g_2)^{-1}.$$

The map $(g, a) \mapsto s(g) \cdot i(a)$ gives an isomorphism from $\mathsf{Inc}(z)$ to E.

1.5. $\mathrm{CExt}(G, -)$ **as a moduli functor. —** For another perspective, fix a locally compact group G. The assignment $A \mapsto \mathrm{CExt}(G, A)$ gives a functor,

$$\mathrm{CExt}(G, -) \colon \mathsf{LCA} \to \mathsf{Ab},$$

where LCA denotes the category of locally compact abelian groups and Ab denotes the category of (abstract) abelian groups. Indeed, we have seen above that $\mathrm{CExt}(G, A)$ has a natural abelian group structure and functoriality comes from