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LARGE KAM TORI FOR PERTURBATIONS OF THE DEFOCUSING NLS EQUATION

by Massimiliano BERTI, Thomas KAPPELER & Riccardo MONTALTO

Abstract. — We prove that small, semi-linear Hamiltonian perturbations of the defocusing nonlinear Schrödinger (dNLS) equation on the circle have an abundance of invariant tori of any size and (finite) dimension which support quasi-periodic solutions. When compared with previous results the novelty consists in considering perturbations which do not satisfy any symmetry condition (they may depend on x in an arbitrary way) and need not be analytic. The main difficulty is posed by pairs of almost resonant dNLS frequencies. The proof is based on the integrability of the dNLS equation, in particular the fact that the nonlinear part of the Birkhoff coordinates is one smoothing. We implement a Newton-Nash-Moser iteration scheme to construct the invariant tori. The key point is the reduction of linearized operators, coming up in the iteration scheme, to 2×2 block diagonal ones with constant coefficients together with sharp asymptotic estimates of their eigenvalues.

Résumé — Dans ce travail on démontre que toutes les perturbations hamiltoniennes de l'équation de Schrödinger nonlinéaire défocalisante (dNLS), qui sont semi-linéaires et suffisamment petites, admettent un grand nombre de tores invariants de taille et de dimension finie arbitrairement grande. Aucune condition de symétrie n'est supposée pour la perturbation et il n'est pas nécessaire qu'elle soit analytique. La difficulté principale est la présence des paires de fréquences de l'équation dNLS qui sont presque résonnantes. La preuve est basée sur l'intégrabilité de l'équation dNLS et en particulier sur le fait, que la partie nonlinéaire des coordonnées de Birkhoff est régularisante. On applique une procédure d'itération de type Newton-Nash-Moser pour construire les tores invariants. Les éléments clé du schéma de la procédure d'itération sont la réduction de certains opérateurs linéaires à des opérateurs, qui sont 2×2 bloc-diagonaux à coéfficients constants, et des estimations asymptotiques précises de leurs valeurs propres.

CONTENTS

CHAPTER 1

INTRODUCTION

Consider the defocusing nonlinear Schrödinger (dNLS) equation in one space dimension

$$\mathrm{i}\partial_t u = -\partial_x^2 u + 2|u|^2 u \tag{1}$$

on the standard Sobolev space $H^\sigma \equiv H^\sigma(\mathbb{T}_1, \mathbb{C})$ of complex valued functions on $\mathbb{T}_1 := \mathbb{R}/\mathbb{Z}$. It is well known that for $\sigma \geq 0$, (1) is wellposed and for $\sigma \geq 1$, it is a Hamiltonian PDE with Poisson bracket and Hamiltonian given by

$$\begin{aligned} \{\mathscr{F}, \mathscr{G}\}(u_1, u_2) &= -\mathrm{i}\int_0^1 (\nabla_u \mathscr{F} \nabla_{\bar{u}} \mathscr{G} - \nabla_{\bar{u}} \mathscr{F} \nabla_u \mathscr{G}) dx, \\ \mathscr{H}^{\mathrm{nls}}(u_1, u_2) &= \int_0^1 (\partial_x u \partial_x \bar{u} + u^2 \bar{u}^2) dx. \end{aligned} \tag{2}$$

Here u_1, u_2 are the real valued functions, defined in terms of u by $u_1 := \sqrt{2}\mathrm{Re}(u)$, $u_2 := -\sqrt{2}\mathrm{Im}(u)$, the L^2-gradients $\nabla_u, \nabla_{\bar{u}}$ are given by $\nabla_u := (\nabla_{u_1} + \mathrm{i}\nabla_{u_2})/\sqrt{2}$, $\nabla_{\bar{u}} := (\nabla_{u_1} - \mathrm{i}\nabla_{u_2})/\sqrt{2}$, and $\mathscr{F}, \mathscr{G}$, viewed as functions of u_1 and u_2, are $\mathscr{C}^1$-smooth, real valued functionals on H^σ with sufficiently regular L^2-gradients. The Hamiltonian vector field corresponding to $\mathscr{H}^{\mathrm{nls}}$ can then be computed to be $-\mathrm{i}\nabla_{\bar{u}}\mathscr{H}^{\mathrm{nls}}$ and when written in Hamiltonian form, Equation (1) becomes $\partial_t u = -\mathrm{i}\nabla_{\bar{u}}\mathscr{H}^{\mathrm{nls}}$. According to **[23]**, (1) is an integrable PDE in the strongest possible sense, meaning that it admits global Birkhoff coordinates on H^σ, $\sigma \in \mathbb{Z}_{\geq 0}$—see Subsection 3.1 for more details. In these coordinates, Equation (1) can be solved by quadrature and the phase space H^σ is the union of compact, connected tori, invariant under the flow of (1). All the solutions are periodic, quasi-periodic or almost periodic in time. These invariant tori are denoted by $\mathscr{T}_I$ where the parameters $I = (I_k)_{k\in\mathbb{Z}}$ are the action variables, which are defined in terms of the Birkhoff coordinates and fill out the whole positive quadrant $\ell_+^{1,2\sigma}$ of the weighted sequence space $\ell^{1,2\sigma} \equiv \ell^{1,2\sigma}(\mathbb{Z}, \mathbb{R})$. The dimension of $\mathscr{T}_I$ coincides with the cardinality $|S|$ of the index set $S \equiv S_I \subseteq \mathbb{Z}$, given by $S = \{k \in \mathbb{Z} \mid I_k > 0\}$. In case $|S| < \infty$, it can be shown that the functions in $\mathscr{T}_I$ are $\mathscr{C}^\infty-$smooth (cf. e.g., **[23]**) and that solutions of (1) with initial data in $\mathscr{T}_I$ wrap around $\mathscr{T}_I$ with speed, defined in terms of the dNLS frequencies $\omega_k^{\mathrm{nls}}(I)$, $k \in S$. They are called S-gap solutions.

Our aim is to prove that for Hamiltonian perturbations

$$\mathrm{i}\partial_t u = -\partial_x^2 u + 2|u|^2 u + \varepsilon f(x, u) \tag{3}$$

of Equation (1), many of these finite dimensional tori persist, provided that ε is sufficiently small. The perturbation f is assumed to be given by $f(x, u) = \nabla_{\bar{u}}\mathscr{P}$ where $\mathscr{P}$ is a real valued Hamiltonian of the form

$$\mathscr{P}(u) = \int_0^1 p(x, u_1(x), u_2(x))dx \tag{4}$$

and p a real valued function

$$p \ : \mathbb{T}_1 \times \mathbb{R}^2 \to \mathbb{R}, \ (x, \zeta_1, \zeta_2) \mapsto p(x, \zeta_1, \zeta_2)$$

which is then related to $f : \mathbb{T}_1 \times \mathbb{C} \to \mathbb{C}$ by the identity, valid for any $\zeta = (\zeta_1 - \mathrm{i}\zeta_2)/\sqrt{2}$ with $\zeta_1, \zeta_2 \in \mathbb{R}$,

$$f(x, \zeta) = \partial_{\bar{\zeta}} p\ (x, \zeta_1, \zeta_2), \qquad \partial_{\bar{\zeta}} := \big(\partial_{\zeta_1} - \mathrm{i}\partial_{\zeta_2}\big)/\sqrt{2}. \tag{5}$$

We assume that f is $\mathcal{C}^{\sigma, s_*}$-smooth, meaning that

$$\partial_x^\alpha \partial_{\zeta_1}^{\beta_1} \partial_{\zeta_2}^{\beta_2} f \in \mathcal{C}(\mathbb{T}_1 \times \mathbb{C}, \mathbb{C}), \quad \forall\, 0 \le \alpha \le \sigma, \quad \forall\, 0 \le \beta_1, \beta_2 \le s_*. \tag{6}$$

In the above formula, $\mathcal{C}(\mathbb{T}_1 \times \mathbb{C}, \mathbb{C}) \equiv \mathcal{C}^0(\mathbb{T}_1 \times \mathbb{C}, \mathbb{C})$ denotes the space of continuous functions defined on $\mathbb{T}_1 \times \mathbb{C}$, with values in $\mathbb{C}$. Note that $f(x, \zeta)$ need not be complex differentiable in ζ. To state our result in detail, introduce for any given subset $S \subseteq \mathbb{Z}$ of finite cardinality $|S| < \infty$, the parameter space

$$\Pi_S := \big\{(\xi_k)_{k\in\mathbb{Z}} \subset \mathbb{R} \,\big|\, \xi_k = 0 \ \forall k \in \mathbb{Z}\backslash S; \ \ \xi_k > 0 \ \forall k \in S\big\},$$

which we identify with $\mathbb{R}^S_{>0}$ formed by the vectors $(\xi_k)_{k\in S}$ with $\xi_k > 0$ for any k in the index set S. Here and sometimes in the sequel, by a slight abuse of terminology, we write $(\xi_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ to denote the real valued sequence, $\mathbb{Z} \to \mathbb{R}$, $k \mapsto \xi_k$. Elements of S are referred to as tangential sites whereas elements of $S^\perp := \mathbb{Z} \setminus S$ are referred to as normal sites. Correspondingly we shall call ω_k^{nls}, $k \in S$, the tangential dNLS frequencies and ω_k^{nls}, $k \in S^\perp$, the normal ones.

By the non-degeneracy property (86) of Proposition 3.3, the action-to-frequency map

$$\omega^S : \Pi_S \to \mathbb{R}^S, \ I \mapsto (\omega_k^{\mathrm{nls}}(I))_{k\in S} \tag{7}$$

is a local diffeomorphism on an open, dense subset of Π_S. Finally, let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Our main result is the following one.

Theorem 1.1. — *Let $\sigma \in \mathbb{Z}_{\ge 4}$ and $S \subset \mathbb{Z}$ with $|S| < \infty$, $0 \in S$, and $-S = S$ be given and assume that $\Pi \subseteq \Pi_S$ is a compact subset of positive Lebesgue measure,* $\mathrm{meas}(\Pi) > 0$, *with the property that the action-to-frequency map $\omega^{\mathrm{nls}} : \Pi \to \mathbb{R}^S$, $I \mapsto (\omega_k^{\mathrm{nls}}(I))_{k\in S}$, is a bi-Lipschitz homeomorphism onto its image Ω. Then there is an integer $s_* > \max\big(\sigma, |S|/2\big)$ so that for any Hamiltonian $\mathscr{P}$ of the form* (4) *with*

$f = \nabla_{\bar{u}}\mathscr{P}$ of class $\mathcal{C}^{\sigma,s_*}$, there exist $\varepsilon_0 > 0$ and $|S|/2 < s < s_*$ so that for any $0 < \varepsilon \leq \varepsilon_0$ the following holds: there exist a closed subset $\Omega_\varepsilon \subseteq \Omega$, satisfying

$$\lim_{\varepsilon \to 0} \frac{\text{meas}(\Omega_\varepsilon)}{\text{meas}(\Omega)} = 1, \tag{8}$$

and a Lipschitz family of maps $\iota_\omega : \mathbb{T}^S \to H^\sigma$, $\omega \in \Omega_\varepsilon$, so that ι_ω are H^s-smooth embeddings with the property that for any initial data $\varphi \in \mathbb{T}^S$, the curves

$$t \mapsto \iota_\omega(\varphi + t\omega)$$

are quasi-periodic solutions of (3)*. The torus described by the map ι_ω is invariant under the flow of the perturbed Hamiltonian $\mathscr{H}^{\text{nls}} + \varepsilon\mathscr{P}$.*

In Theorem 4.1 we will show in addition that, for $\omega \in \Omega_\varepsilon$, the distance of the invariant torus $\iota_\omega(\mathbb{T}^S)$ to the unperturbed torus $\mathscr{T}_{\xi(\omega)}$ is of the order $O\big(\varepsilon\gamma^{-2}\big)$ where $0 < \gamma < 1$ is the constant appearing in the diophantine condition of ω introduced in (22). Here $\xi(\omega)$ denotes the element in Π, corresponding to ω by the action-to-frequency map defined in (7). Expressing Equation (3) in suitable coordinates, one sees that actually the distance of the invariant torus to the unperturbed one is $O(\varepsilon\gamma^{-1})$, see Corollary 8.3. Note that the frequency vector ω of the quasi-periodic solution $\iota_\omega(\varphi + t\omega)$ of (3) is the same as the one of the quasi-periodic solutions on the invariant torus $\mathscr{T}_{\xi(\omega)}$ of (1).

Comments

1. Using a covering argument one can show that Theorem 1.1 actually holds for any compact subset $\Pi \subseteq \Pi_S$ with $\text{meas}(\Pi) > 0$. See the comment after Theorem 4.1.
2. In Theorem 9.1 we prove that for some $\nu > 0$, $\text{meas}(\Omega \setminus \Omega_\varepsilon) = O(\varepsilon^\nu)$ as $\varepsilon \to 0$.
3. The assumption $0 \in S$ and $S = -S$ are introduced just for simplicity, so that all elements in the complement $\mathbb{Z} \setminus S$ of S come in pairs, so that in the reduction procedure in Section 7 we only have to deal with 2×2 blocks.
4. By (6) the perturbation f is assumed to be $\mathcal{C}^{\sigma,s_*}$-smooth where a lower bound for s_* is given in Theorem 8.1 (Nash-Moser). Note that the regularity with respect to the space variable is just $\sigma \in \mathbb{Z}_{\geq 4}$. No special effort has been made to get optimal lower bounds for s_* and σ.

Outline of the proof of Theorem 1.1. — The starting point of our proof is to write the perturbed dNLS Equation (3), which is a Hamiltonian PDE with Hamiltonian $\mathscr{H}^{\text{nls}} + \varepsilon\mathscr{P}$, in complex Birkhoff coordinates $(w_k)_{k\in\mathbb{Z}}$, which are briefly reviewed in Subsection 3.1. The unperturbed dNLS-Hamiltonian $\mathscr{H}^{\text{nls}}$, expressed in these coordinates, is a real analytic function H^{nls} of the actions $I_k = w_k\bar{w}_k$, $k \in \mathbb{Z}$, and the dNLS frequencies ω_k^{nls} are given by

$$\omega_k^{\text{nls}} = \partial_{I_k} H^{\text{nls}}, \quad k \in \mathbb{Z}.$$

Denoting by P the Hamiltonian $\mathscr{P}$, expressed in these coordinates, Equation (3) then becomes the following infinite dimensional Hamiltonian system

$$\mathrm{i}\dot{w}_k = \omega_k^{\text{nls}} w_k + \varepsilon\partial_{\bar{w}_k} P, \ k \in \mathbb{Z}, \tag{9}$$