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FEYNMAN-KAC FORMULAS FOR THE ULTRA-VIOLET RENORMALIZED NELSON MODEL

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Abstract. — We derive Feynman-Kac formulas for the ultra-violet renormalized Nelson Hamiltonian with a Kato decomposable external potential and for corresponding fiber Hamiltonians in the translation invariant case. We simultaneously treat massive and massless bosons. Furthermore, we present a non-perturbative construction of a renormalized Nelson Hamiltonian in a non-Fock representation defined as the generator of a corresponding Feynman-Kac semi-group. Our novel analysis of the vacuum expectation of the Feynman-Kac integrands shows that, if the external potential and the Pauli-principle are dropped, then the spectrum of the N-particle renormalized Nelson Hamiltonian is bounded from below by some negative universal constant times $\mathfrak{g}^4 N^3$, for all values of the coupling constant $\mathfrak{g}$. A variational argument also yields an upper bound of the same form for large $\mathfrak{g}^2 N$. We further verify that the semi-groups generated by the ultra-violet renormalized Nelson Hamiltonian and its non-Fock version are positivity improving with respect to a natural self-dual cone, if the Pauli principle is ignored. In another application we discuss continuity properties of elements in the range of the semi-group of the renormalized Nelson Hamiltonian.

***Résumé* (*Formules de Feynman-Kac pour le modèle de Nelson ultra-violet renormalisée*)**

On s'intéresse à la dérivation des formules de Feynman-Kac pour l'Hamiltonien du modèle de Nelson ultra-violet renormalisée avec potentiel Kato-décomposable, et pour les fibré Hamiltoniens correspondants dans le cas de l'invariance par translation. On traite simultanément des bosons lourds et sans masse. On présente également une construction non perturbative de l'Hamiltonien de Nelson renormalisée dans une représentation de *non-Fock*, définie comme étant le générateur du semi-groupe de Feyman-Kac associé. La nouvelle approche de l'analyse des valeurs moyennes associées au vide des intégrandes de Feynman-Kac montre que, en l'absence de potentiel externe et dans le cas où le principe de Pauli est ignoré, le spectre de l'Hamiltonien du modèle de Nelson avec N particules est minoré par une constante négative universelle multipliée par $\mathfrak{g}^4 N^3$, pour n'importe quelle valeur de la constante de couplage $\mathfrak{g}$. Un argument variationnel permet également d'obtenir une majoration faisant intervenir une quantité analogue, pour de grandes valeurs de $\mathfrak{g}^2 N$. On vérifie de plus que le semi-groupe généré par l'Hamiltonien du modèle de Nelson ultra-violet renormalisée améliore la positivité par rapport à un cône auto-dual naturel, pourvu que le principe de Pauli est exclut. Une partie de l'étude s'intéresse également aux propriétés de continuité des éléments de l'image du semi-groupe associé à l'Hamiltonien de Nelson renormalisée.

CONTENTS

CHAPTER 1

INTRODUCTION

More than half a century ago, Edward Nelson studied the renormalization theory of a model for a conserved number of non-relativistic scalar matter particles interacting with a quantized radiation field comprised of relativistic scalar bosons. This model is *a priori* given by a heuristic Hamiltonian equal to the sum of the Schrödinger operator for the matter particles, the radiation field energy operator, and a field operator describing the interaction between the matter particles and the radiation field. This heuristic expression is, however, mathematically ill-defined because the physically relevant choice of the interaction kernel determining the field operator is not a square-integrable function of the boson modes. Hence, one starts out by introducing an artificial ultra-violet cut-off rendering this kernel function square-integrable and the Hamiltonian well-defined. The question, then, is whether the so-obtained ultraviolet regularized Nelson Hamiltonians converge in a suitable sense as the cut-off is removed, possibly after adding cut-off dependent energy shifts that would not harm physical interpretations. Nelson approached this mathematical problem by probabilistic methods in [**48**] and by operator theoretic arguments in [**49**].

In his earlier probabilistic investigation Nelson eventually considered the matrix elements of the unitary groups generated by the regularized Hamiltonians with an explicitly given energy renormalization added and proved that, for strictly positive boson masses and fixed time parameters, these matrix elements are convergent as the cut-off is removed, in the weak-$*$ sense as bounded measurable functions of the particle mass. While he knew that these limits are non-trivial, he could not yet decide whether they define a new unitary group or not. This was clarified in his second work cited where, after adding the energy renormalization and applying unitary Gross transformations, he obtained a sequence of Hamiltonians converging in the norm resolvent sense. As the Gross transformations converge strongly, this implied strong resolvent convergence of the original regularized operators plus the energy shifts towards a renormalized Nelson Hamiltonian.

Ealier results on Nelson's renormalized operator. — Given that the two aforementioned papers of Nelson date back to 1964 the number of mathematical articles explicitly addressing properties of his renormalized model is not very large, whence we give a brief, essentially chronological survey in what follows.

The first results following [**48**], [**49**] are the construction of asymptotic fields for massive bosons by Høegh-Krohn [**35**] and of renormalized fiber Hamiltonians in the translation-invariant case by Cannon [**12**], who adapted the procedure in [**49**]. Cannon also proved the existence of non-relativistic Wightman distributions and, for a sufficiently weak matter-radiation coupling, the existence of dressed one-particle states as well as analyticity of the corresponding energies and eigenvectors. Cannon's smallness assumptions on the coupling depend on the strictly positive boson mass he was cosidering. His results were pushed forward by Fröhlich [**22**] who gave non-perturbative proofs of Cannon's results, that hold for any strictly positive boson mass or, alternatively, infra-red cut-off, irrespective of the value of the coupling constant. In this article Fröhlich also found a rich class of Hilbert spaces, including examples of von Neumann's incomplete direct product spaces, on which the renormalization procedure of [**49**] can be implemented. Fröhlich [**21**] employed his results to discuss the infra-red problem and aspects of scattering theory for a class of models containing the Nelson model. In particular, for vanishing boson mass and without any cut-offs, he constructed coherent infra-red representation spaces which are attached to total momenta of the matter-radiation system and contain dressed one-particle states that are ground states of, roughly speaking, certain non-Fock versions of the renormalized fiber Hamiltonians. He also proved the absence of dressed one-particle states in the original Fock space for vanishing boson mass, a phenomenon known as infra-particle situation.

After a gap of more than twenty-five years in the mathematical literature on the renormalized Nelson Hamiltonian, its spectral and scattering theory in a confining potential and for massive bosons has been worked out by Ammari [**3**], who proved a HVZ theorem, positive commutator estimates, the existence of asymptotic fields, propagation estimates, and asymptotic completeness. A few years later, Hirokawa *et al.* [**33**] considered a system of two particles with charges of equal sign, one of them static, interacting via a linearly coupled massless boson field. After applying a Gross transformation to the corresponding ultra-violet regularized Hamiltonian they found a Hamiltonian for one particle coupled to the radiation field with an additional attractive potential playing the role of an effective interaction between the two particles. The Gross transformation is actually infra-red singular for zero boson mass. Thus, an artificial infra-red cut-off is included in its definition. By improving some of Nelson's [**49**] relative bounds so as to cover massless bosons, Hirokawa *et al.* removed both the ultra-violet and infra-red cut-offs in their Gross transformed Hamiltonian and obtained, for sufficiently small coupling constants, a self-adjoint operator called renormalized Nelson Hamiltonian in the associated non-Fock representation. The latter turned out to have a ground state eigenvector. Employing this result, Hainzl *et al.* [**32**] established a formula for the first radiative correction to the binding energy of this interacting two-particle system. Not long after these developments, a paper of Ginibre *et al.* [**25**] appeared concerning a certain partially classical limit of the Nelson model for any non-negative boson mass and without cut-offs.

The investigation of the infra-particle nature of the massless Nelson model without cut-offs has been revisited more recently by Bachmann *et al.* [**6**] employing Pizzo's iterative analytic perturbation theory. While their results hold for sufficiently small coupling only, they provide more detailed control on the mass-shell and the dressed one-particle states than earlier results.

Nelson's operator theoretic renormalization procedure [**49**] has also been implemented on static Lorentzian manifolds and for position-dependent boson masses by Gérard *et al.* [**23**]. Comparatively recently, Nelson's [**48**] earlier approach has been revived as well by Gubinelli *et al.* [**29**], who succeeded by *probabilistic* arguments to verify strong convergence of the semi-group as an ultra-violet regularization is removed in a Nelson Hamiltonian for massive bosons. In the same paper, Gubinelli *et al.* also computed effective potentials in the weak coupling limit of the renormalized theory. Hiroshima [**34**] treated infra-red cut-off fiber Hamiltonians along the same lines as well.

Ammari and Falconi [**4**] proved a Bohr correspondence principle showing that, in a classical limit, the time evolution of quantum states generated by a renormalized Nelson Hamiltonian for massive bosons converges to the push-forward of a Wigner measure under the dynamics of a nonlinear Schrödinger-Klein-Gordon system. They also explored the idea to carry through a renormalization procedure on the classical level and to Wick quantize the result afterwards, which leads to the same renormalized operator in the Nelson model.

Bley and Thomas [**7**], [**8**], [**10**] recently developed a general new method to bound a class of exponential moments that often arise when functional integration techniques are applied in non-relativistic quantum (field) theory. Applied to the renormalized Nelson Hamiltonian, with non-negative boson mass and vanishing exterior potential, this method yields a lower bound on its spectrum of the form

$$-c\mathfrak{g}^4N^3\big(1\vee\ln^2([1\vee\mathfrak{g}^2]N)\big),$$

(see [**8**]), where N is the number of matter particles and the modulus of the coupling constant $\mathfrak{g}$ is either assumed to be sufficiently large or sufficiently small. Here we should add that, as we shall do in the present work, Bley fixes the explicit energy counter terms in the renormalization procedure, which are proportional to $\mathfrak{g}^2$, in such a way that no contribution of order $\mathfrak{g}^2$ shows up in his lower bound for the renormalized operator. This differs from the convention in [**49**]. Using his bound, Bley [**9**] also provided a non-binding condition in the massless Nelson model for two matter particles, whose effective attraction mediated by the radiation field is compensated for by a repulsive Coulomb interaction.

Note added in proof. — Some information on the domain of the renormalized Nelson operator has been obtained by Griesemer and Wünsch [**27**]. In fact, they showed that the only vector belonging both to the form domain of the renormalized Nelson operator and the form domains of the ultra-violet regularized operators is the zero vector. Furthermore, in a recent preprint, Miyao [**46**] proved that the semigroups generated by the fiber Hamiltonians in the translation invariant renormalized Nelson model are ergodic, provided that elements of the usual Fock space are called positive, if all their n-particle components are a.e. positive functions.

We restricted the above summary to articles explicitly containing theorems on the renormalized Nelson model, as an account on the numerous mathematical papers devoted to ultra-violet regularized Nelson Hamiltonians would be far too space-consuming. For a general introduction to the model and more references the reader can consult, *e.g.*, the textbook [**40**]. A renormalization of a translation-invariant Nelson type model for a *relativistic* scalar matter particle interacting with a massive boson field [**28**] actually leads to a theory with a flat mass shell [**15**].

Description of results. — The first main result of the present book is a novel Feynman-Kac formula for the renormalized Nelson Hamiltonian for N matter particles in a Kato decomposable external potential V and for non-negative boson masses. Denoting the latter operator by $H^V_{N,\infty}$ it reads

$$(1.1)\qquad \begin{cases} (\mathrm{e}^{-tH^V_{N,\infty}}\Psi)(\underline{x}) = \mathbb{E}\big[W^V_{\infty,t}(\underline{x})^*\Psi(\underline{x}+\underline{b}_t)\big], \quad \text{a.e.,} \\ W^V_{\infty,t}(\underline{x})^* = \mathrm{e}^{u^N_{\infty,t}(\underline{x})-\int_0^t V(\underline{x}+\underline{b}_s)\mathrm{d}s}F_{0,t/2}\big(-U^{N,-}_{\infty,t}(\underline{x})\big)F_{0,t/2}\big(-U^{N,+}_{\infty,t}(\underline{x})\big)^*, \end{cases}$$

for every $t>0$, where, in standard notation recalled later on,

$$F_{0,s}(f) := \sum_{n=0}^{\infty}\frac{a^\dagger(f)^n}{n!}\,\mathrm{e}^{-s\,\mathrm{d}\Gamma(\omega)}, \quad s>0.$$

In (1.1), Ψ is a Fock space-valued square-integrable function of $\underline{x}\in\mathbb{R}^{3N}$ and $\underline{b}$ is a $3N$-dimensional Brownian motion. The real-valued stochastic process $u^N_{\infty,t}(\underline{x})$ is called the complex action following Feynman [**19**] and the $U^{N,\pm}_{\infty,t}(\underline{x})$ are continuous adapted stochastic processes with values in the one-boson Hilbert space $\mathfrak{h}:=L^2(\mathbb{R}^3)$. The series defining $F_{0,s}(f)$ converges in the Fock space operator norm and defines an analytic function of $f\in\mathfrak{h}$.

For ultra-violet regularized Nelson Hamiltonians, the special form (1.1) of the Feynman-Kac formula appeared in [**30**]. We shall re-prove it to make this book essentially self-contained and to demonstrate that the Nelson model admits a simpler proof than the models in [**30**] which in general involve minimally coupled fields as well.

In fact, our derivation of (1.1) consists in implementing a new renormalization procedure on the level of semi-groups in the spirit of [**29**], [**48**] and re-defining $H_{N,\infty}^{V}$ as the generator of the semi-group given by the right hand sides in (1.1). We shall actually observe *norm* convergence of semi-groups with hardly any technical restriction on the details of the ultra-violet regularization; see also [**3**] as well as Theorem 2.4 and the remarks following it. Our definition of $H_{N,\infty}^{V}$ is manifestly independent of the choice of any cut-off function, purely and simply as this is the case for the right hand sides in (1.1). With comparatively little extra work we shall also derive new Feynman-Kac formulas for the renormalized Nelson Hamiltonian in the non-Fock representation considered in [**32**] and for fiber Hamiltonians in the translation-invariant renormalized Nelson model. In particular, we shall provide the first non-perturbative construction of a renormalized Nelson Hamiltonian in a non-Fock representation.

The crucial point about the Feynman-Kac representation (1.1) is that it provides a fairly simple and tractable formula for a well-defined *Fock space operator-valued process* $W_{\infty}^{V}(\underline{x})$ in the Feynman-Kac integrand and can be applied to *every* element Ψ of the Hilbert space for the whole system. While Nelson and Gubinelli *et al.* have Feynman-Kac type representations of expectation values with respect to vectors in certain total subsets of the Hilbert space (involving suitable finite particle states [**48**] or coherent states [**29**] in Fock space), the merit of writing the Feynman-Kac formula in the form (1.1) is that it allows to first find explicit expressions for $U_{\infty,t}^{N,\pm}(\underline{x})$ containing well-defined $\mathfrak{h}$-valued stochastic integrals and then to derive *operator-norm bounds* on $W_{\infty}^{V}(\underline{x})$ with finite moments of any order. Furthermore, our formulas permit to verify a Markov property of the Feynman-Kac integrand. In particular, we can work out basic features of a semi-group theory in Fock space-valued L^p-spaces in the spirit of [**13**], [**56**]. Along the way we further present a new method to bound the exponential moments of the complex action $u_{\infty,t}^{N}(\underline{x})$ which eventually leads to the improved lower bound

$$\inf \sigma(H_{N,\infty}^{0}) \geqslant -c\mathfrak{g}^4 N^3, \tag{1.2}$$

valid for *all* $\mathfrak{g}$ and N with a universal constant $c > 0$; see the introduction to Section 4.3 for more remarks on this new method and a discussion of earlier results [**7**], [**8**], [**10**], [**29**], [**48**]. (Here we ignore that the matter particles are supposed to be fermions, *i.e.*, the Pauli principle is neither taken into account here nor in [**7**], [**8**], [**10**], [**29**], [**48**].) We shall employ a novel bound on an ultra-violet part of $u_{\infty,t}^{N}(\underline{x})$ together with a more standard trial function argument to derive the upper bound in

$$\begin{aligned} -(16\pi)^2\mathfrak{g}^4N^3 - c'\mathfrak{g}^2N^2 &\leqslant \inf\sigma(H_{N,\infty}^{0}) \\ &\leqslant 8\pi^4 E_{\mathrm{P}}\mathfrak{g}^4N^3 + c''\big(1+\mu+\ln(\mathfrak{g}^2N)\big)\mathfrak{g}^2N^2, \quad \text{provided that } \mathfrak{g}^2N \geqslant c. \end{aligned} \tag{1.3}$$

Here $\mu \geqslant 0$ is the boson mass, $E_{\mathrm{P}} < 0$ is the Pekar energy, and $c, c', c'' > 0$ are universal constants. (With $\mathfrak{g}_{\mathrm{N}}$ denoting the coupling constant in Nelson's articles [**48**], [**49**], we have the relation $2^{1/2}(2\pi)^{3/2}\mathfrak{g} = \mathfrak{g}_{\mathrm{N}}$.) The leading behavior $\propto -\mathfrak{g}^4N^3$ in (1.2) and (1.3) is familiar from the closely related Fröhlich polaron model [**7**], [**8**], [**39**], which can be renormalized as in [**49**] even without introducing energy counter terms. (If a sufficiently strong electrostatic Coulomb repulsion between the matter particles is taken into account, then one actually observes thermodynamic stability, *i.e.*, a behavior of the minimal energy proportional to $-N$ in the Fröhlich polaron model without restriction to symmetry subspaces [**20**]. For sufficiently weak electrostatic repulsion, the minimal energy of *fermionic* multi-polaron systems behaves like $-N^{7/3}$, see [**26**].) The work on the polaron model [**39**] suggests that $8\pi^4 E_{\mathrm{P}}$ should in fact be the correct leading coefficient in (1.3). Numerics shows (see [**24**]) that

$$E_{\mathrm{P}} = -0.10851\ldots,$$

whence the leading coefficient in the lower bound in (1.3) is presumably too large by the factor $32/\pi^2|E_{\mathrm{P}}| < 30$. Getting rid of this artifact is, however, beyond the scope of this book.

Finally, we present two applications of the new formula (1.1). First, we shall fill a gap left open in the earlier literature by proving that the semi-groups generated by the renormalized Nelson Hamiltonian and its non-Fock version are positivity improving at positive times with respect to a natural convex cone. In the non-Fock case this result was explicitly mentioned as an open problem in [**33**, §10] and it entails uniqueness and strict positivity of the ground state eigenvector found there. As already observed in [**43**] the ergodicity of the semi-groups follows easily from the structure of the integrand in (1.1) and standard tools associated with Perron-Frobenius type arguments in quantum field theory; see, *e.g.*, [**18**], [**57**]. In the second application we employ some results of [**43**] on ultra-violet regularized operators to discuss the continuous dependence of the right hand side in the first line of (1.1) on $\underline{x}$, $\mathfrak{g}$, and V.

Organization and general notation

The remaining part of this book is structured as follows

- ▹ In Chapter 2 we introduce some basic notation and give a precise definition of Nelson's model.
- ▹ In Chapter 3 we shall analyze certain $\underline{x}$-independent one-particle versions of $U^{N,\pm}_{\infty,t}(\underline{x})$ and eventually define the latter two processes.
- ▹ Chapter 4 is devoted to the complex action $u^{N}_{\infty,t}(\underline{x})$.
- ▹ In Chapter 5 we work out the semi-group properties between Fock space-valued L^p-spaces including the norm convergence of semi-groups, as the ultra-violet cut-off is removed.

- At the end of Chapter 5 we establish the above Feynman-Kac formula and (re-)define the renormalized Nelson Hamiltonian; see Theorem 5.13 and Definition 5.14. (Our version of Nelson's theorem is also anticipated in Theorem 2.4.) The lower bound (1.2) is obtained in Corollary 5.16.
- The Feynman-Kac formulas in the non-Fock representation and for the fiber Hamiltonians are derived in Chapter 6 and Chapter 7, respectively.
- The positivity improving and continuity properties alluded to above as well as the bounds in (1.3) are proved in Chapter 8.
- The main text is followed by three appendices presenting well-known material on the Kolmogorov test lemma, exponential moment bounds for sums of pair potentials (see also [**8**]), and a general formula for the infimum of a spectrum.

Some general notation

- The characteristic function of a set A is denoted by $1\!\!1_A$.
- We abbreviate

$$a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}, \quad \text{for } a, b \in \mathbb{R}.$$

- The Borel σ-algebra of a topological space $\mathcal{T}$ is denoted by $\mathfrak{B}(\mathcal{T})$.
- The Lebesgue-Borel measure on $\mathbb{R}^n$ is denoted by λ^n and, as usual, we shall write

$$\mathrm{d}t := \mathrm{d}\lambda^1(t), \quad \mathrm{d}\boldsymbol{x} := \mathrm{d}\lambda^3(\boldsymbol{x}), \quad \textit{etc.},$$

  if a symbol t, $\boldsymbol{x}$, *etc.*, for the integration variable is specified.
- The set of bounded operators on a Banach space $\mathcal{X}$ is denoted by $\mathscr{B}(\mathcal{X})$.
- The symbols $\mathscr{D}(T)$ and $\mathbb{Q}(T)$ stand for the domain and form domain, respectively, of a suitable linear operator T.
- The spectrum of a self-adjoint operator T in a Hilbert space is denoted by $\sigma(T)$.
- The symbols $c_{a,b,\dots}, c'_{a,b,\dots}, \dots$ denote non-negative constants that depend solely on the quantities displayed in their subscripts (if any). Their values might change from one estimate to another.