Séminaires & Congrès

COLLECTION SMF

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CONTROL AND STABILIZATION OF PARTIAL DIFFERENTIAL EQUATIONS

Numéro 29

Kaïs Ammari, ed.

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Séminaires & Congrès
29, 2015, p. 1–50

STABILIZATION OF NAVIER-STOKES EQUATION

by

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Abstract. — We survey here a few recent results and methods to stabilization of equilibrium solutions to Navier-Stokes in 2-D and 3-D.

1. The stabilization problem

Consider the Navier-Stokes equation in a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$,

$$(1.1)\qquad \begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y = f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O},\\ &\nabla \cdot y = 0 && \text{in } \mathbb{R}^+ \times \mathcal{O},\\ &y = 0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O},\\ &y(0) = y_0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \end{aligned}$$

where $f_e \in (L^2(\mathcal{O}))^d$, $\nabla \cdot f_e = 0$, $f_e \cdot n = 0$.

Here n is the unit normal and is directed toward the exterior of $\partial\mathcal{O}$.

Let $y_e \in (H^2(\mathcal{O}))^d$ be an equilibrium solution to (1.1), that is,

$$(1.2)\qquad \begin{aligned} &-\nu\Delta y_e + (y_e \cdot \nabla)y_e = f_e + \nabla p_e \quad \text{in } \mathcal{O},\\ &\nabla \cdot y_e = 0 \ \text{ in } \mathcal{O}, \qquad y_e = 0 \quad \text{on } \partial\mathcal{O}. \end{aligned}$$

1.1. Internal stabilization. — Let $\mathcal{O}_0 \subset \mathcal{O}$ be an open subdomain of $\mathcal{O}$ and consider the controlled system associated with (1.1)

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = f_e + \nabla p + \mathbb{1}_{\mathcal{O}_0}u \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ &\nabla\cdot y \quad \text{in } \mathbb{R}^+\times\mathcal{O}; \qquad y=0 \quad \text{on } \mathbb{R}^+\times\partial\mathcal{O}, \\ &y(0)=y_0 \quad \text{in } \mathcal{O}, \end{aligned} \tag{1.3}$$

where the controller u is in $L^2(0,\infty;(L^2(\mathcal{O}))^d)$.

Problem 1.1. — *Find the controller u in feedback form, that is $u(t)=\phi(y(t)-y_e)$ such that the solution to the corresponding solution y to the closed loop system* (1.3) *satisfies for all y_0 in a neighborhood of y_e*

$$\|y(t)-y_e\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y_0-y_e\|_{(L^2(\mathcal{O}))^d}, \ \forall t\ge 0, \tag{1.4}$$

where $\gamma>0$.

If we set $y-y_e\to y$, Problem 1.1 reduces to find $u=\phi(y)$ such that the solution y to the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y+(y\cdot\nabla)y+(y_e\cdot\nabla)y+(y\cdot\nabla)y_e=\nabla p+\mathbb{1}_{\mathcal{O}_0}u, \quad t\ge 0, \\ &\nabla\cdot y=0 \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ &y=0 \quad \text{on } \mathbb{R}^+\times\partial\mathcal{O}, \\ &y(0,x)=y_0(x)-y_e(x)=y^0(x), \ x\in\mathcal{O}. \end{aligned} \tag{1.5}$$

satisfies

$$\|y(t)\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y^0\|_{(L^2(\mathcal{O}))^d}, \quad \forall t\ge 0. \tag{1.6}$$

We use the standard formalism to represent the Navier-Stokes equations as infinite-dimensional differential equations (see, e.g., [**9**], [**20**], [**21**]). That is we set

$$\begin{aligned} H &= \{y\in(L^2(\mathcal{O}))^d; \nabla\cdot y=0 \ \text{in } \mathcal{O}, \ y\cdot n=0 \ \text{on } \partial\mathcal{O}\}, \\ Ay &= -P(\Delta y), \ \forall y\in D(A)=y\in H\cap(H_0^1(\mathcal{O}))^d\cap(H^2(\mathcal{O}))^d, \\ A_0y &= P((y_e\cdot\nabla)y+(y\cdot\nabla)y_e), D(A_0)=H\cap(H_0^1(\mathcal{O}))^d, \\ By &= P((y\cdot\nabla)y), \end{aligned}$$

where $P:(L^2(\mathcal{O}))^d\to H$ is the Leray projector.

We may rewrite (1.5) as

$$\frac{dy}{dt}+\nu Ay+A_0y+By=P(\mathbb{1}_{\mathcal{O}_0}u), \ t\ge 0, \quad y(0)=y^0, \tag{1.7}$$

or, in a more compact form,

$$\frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0}u),\ t \geq 0, \quad y(0) = y^0, \tag{1.8}$$

where $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is the so called Oseen-Stokes operator

$$\mathcal{A} = \nu A + A_0, \quad D(\mathcal{A}) = D(A). \tag{1.9}$$

Then the *internal stabilization problem* reduces to find a feedback controller $u = \phi(y)$ such that the corresponding solution y to (1.8), that is,

$$\frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0}\phi(y)),\ \forall t \geq 0, \quad y(0) = y_0, \tag{1.10}$$

satisfies

$$|y\tau|_H \leq Ce^{-\gamma t}|y_0|_H,\ \forall t \geq 0, \tag{1.11}$$

for $\gamma > 0$ and all y_0 in a neighborhood of the origin. Here and everywhere in the following, $|\cdot|_H$ is the norm of the space H and $(\cdot,\cdot)_H$ is the corresponding scalar product.

1.2. Boundary stabilization. — Consider the boundary control system associated with (1.1)

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ &\nabla \cdot y = 0 && \text{on } \mathbb{R}^+ \times \mathcal{O}, \quad y = u \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ &y(0) = y_0 && \text{in } \mathcal{O}. \end{aligned} \tag{1.12}$$

Problem 1.2. — *Find a boundary controller u in the feedback form $u = \psi(y - y_e)$ such that the corresponding solution y to (1.5) satisfies (1.4) for all y_0 in a neighborhood of y_e.*

Equivalently, the solution y to

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + (y\cdot\nabla)y_e + (y_e\cdot\nabla)y = \nabla p \ \text{ in } \mathbb{R}^+\times\mathcal{O}, \\ &y = u \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \quad y(0) = y^0 - y_e, \\ &\nabla \cdot y = 0 \ \text{ on } \mathbb{R}^+ \times \mathcal{O}, \end{aligned} \tag{1.13}$$

where $u = \psi(y)$, satisfies (1.4).

If u is tangential, that is $u \cdot n = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$, then the stabilization is said to be *tangential* while, if $u \cdot \tau = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$ (where τ is the tangent vector to $\partial\mathcal{O}$), the stabilization is called *normal.*

Denote by $D : (L^2(\partial\mathcal{O}))^d \to H$ the Dirichlet map defined by

$$-\nu\Delta(Du) + (y_e\cdot\nabla)Du + (Du\cdot\nabla)y_e + kDu = \nabla p \ \text{ in } \mathcal{O}, \qquad Du = u \ \text{ on } \partial\mathcal{O}, \tag{1.14}$$

where $k > 0$ is sufficiently large but fixed.

It turns out that D is well defined on the space of all $u \in (L^2(\partial\mathcal{O}))^d$ such that $u\cdot n = 0$ on $\partial\mathcal{O}$ and that D is continuous from $(H^s(\partial\mathcal{O}))^d \to (H^{s+\frac{1}{2}}(\mathcal{O}))^d \cap H$ if $s \geq \frac{1}{2}$. (See Theorem A.2.1 in [**1**].) Then (1.13) reduces to

$$\frac{dy}{dt} + \mathcal{A}(y - Du) + By = kDu,\ t \geq 0, \quad y(0) = y^0. \tag{1.15}$$

If we denote by $\widetilde{\mathcal{A}}$ the extension, by transposition, $\widetilde{\mathcal{A}} : H \to (D(\mathcal{A}^*))'$ with respect to H as pivot space of the original operator $\mathcal{A}$, that is, $(\widetilde{\mathcal{A}}y, z) = (y, \mathcal{A}^* z)$, for all $z \in D(\mathcal{A})$, we can write (1.15) as

$$\frac{dy}{dt} + \widetilde{\mathcal{A}}y + By = kDu + \widetilde{\mathcal{A}}Du,\ t \geq 0, \quad y(0) = y^0, \tag{1.16}$$

and so, the tangential stabilization problem reduces to find a feedback controller $u = \psi(y)$ such that the solution y to (1.16) satisfies (1.4) for all y in a neighborhood of the origin.

It is obvious that the solution y to the Cauchy problem is taken here in a mild sense

$$y\tau = e^{-\mathcal{A}t}y^0 - \int_0^t e^{-\widetilde{\mathcal{A}}(t-s)}(By(s) + kDu + \widetilde{\mathcal{A}}Du(s))ds, \quad t \geq 0. \tag{1.17}$$

Of course, if $\frac{d}{dt}\,Du \in L^2_{\mathrm{loc}}(0,\infty; H)$, we may rewrite (1.17) as

$$\begin{aligned} y\tau =&\ Du\tau + e^{-\mathcal{A}t}(y^0 - Du(0)) \\ &- \int_0^t e^{-\mathcal{A}(t-s)}\Big(By(s) + kDu(s) - \frac{d}{ds}\,Du(s)\Big)ds, \quad \forall t \geq 0. \end{aligned} \tag{1.18}$$

The functional representation of system (1.13) with normal boundary controller is a more delicate problem.

1.3. Main results

***Theorem 1.3* (Barbu & Triggiani 2004).** — *There is a feedback controller*

$$u = \sum_{i=1}^{M} (R(y - y_e), \psi_i)_{(L^2(\mathcal{O}_0))^d}\psi_i, \quad R \in (L^2(\mathcal{O})), \tag{1.19}$$

which stabilizes exponentially y_e for

$$\|y_0 - y_e\|_W \leq \rho, \quad W = (H^{\frac{1}{2}}(\mathcal{O}))^d.$$

Here M^* is dependent of the multiplicity of eigenvalues λ_j of the Oseen-Stokes operator $\mathrm{Re}\,\lambda_j \leq 0$, *$j = 1, \ldots, N$. The functions ψ_j are linear combinations of eigenfunctions φ_j^*.*