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STABILIZATION OF SECOND ORDER EVOLUTION EQUATIONS WITH UNBOUNDED FEEDBACK DELAY

by

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Abstract. — We will consider abstract second order evolution equations with unbounded feedback with delay. Existence results will be obtained under some realistic assumptions. Sufficient and explicit conditions will be derived that guarantee the exponential or polynomial stability. Some illustrative examples that enter into our abstract framework will be presented.

1. Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and associated norm $\|u\|_H = (u,u)^{1/2}$. Let $A : D(A) \to H$ be a self-adjoint positive operator with a compact inverse in H, and set $V := D(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$, with inner product

$$(u,v)_V := (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)_H, \quad \forall u \in V,$$

and norm $\|u\|_V = \|A^{\frac{1}{2}}u\|_H, \forall u, v \in V$. For $i = 1, 2$, let U_i be a real Hilbert space (identified to its dual space) with inner product $(\cdot,\cdot)_{U_i}$ and associated norm $\|v\|_{U_i} = (v,v)_{U_i}^{1/2}$, and let $B_i \in \mathcal{L}(U_i, V')$.

A commonly used operator A is as follows: Let $V \subset H$ be another real Hilbert space with inner product $(\cdot,\cdot)_V$ and associated norm $\|u\|_V = (u,u)_V^{1/2}$. We assume that V is dense in H and is compactly embedded into H. Let be given a bilinear, symmetric and continuous form $a : V \times V \to \mathbb{R}$ such that a is coercive:

$$\exists \alpha > 0 : a(v,v) \geq \alpha \|v\|_V^2, \quad \forall v \in V.$$

We define

$$\begin{cases} D(A) = \{u \in V : \exists f \in H : a(u,v) = (f,v),\ \forall v \in V\}, \\ Au = f, \quad \forall u \in D(A). \end{cases}$$

A is called the *Friedrichs* extension of the triple (H, V, a). It is well known that A is a self-adjoint positive operator with a compact inverse in H such that $V := D(A^{\frac{1}{2}})$.

We consider the closed loop system

$$(1.1)\qquad \begin{cases} u_{tt}(t)+Au(t)+B_1B_1^*u_t(t)+B_2B_2^*u_t(t-\tau)=0,\ t>0\\ u(0)=u_0,\ u_t(0)=u_1,\ B_2^*u_t(t-\tau)=f^0(t-\tau),\ 0<t<\tau, \end{cases}$$

where τ is a positive constant which represents the delay, $u:[0,\infty)\to H$ is the state of the system.

Our main goal is to prove existence and stability results for problem (1.1) under realistic assumptions. Further in order to show the usefulness of our approach, we give some examples where our abstract framework can be applied. This approach goes back to [**9**] but a similar one can be found in [**1**].

Example 1.1. — *(bounded feedbacks). — The 1-d wave equation with internal bounded feedbacks*
(1.2)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t)-\dfrac{\partial^2 u}{\partial x^2}(x,t)+\alpha_1\dfrac{\partial u}{\partial t}(x,t)+\alpha_2\dfrac{\partial u}{\partial t}(x,\,t-\tau)=0, \quad 0<x<1,\,t>0\\ u(0,\,t)=u(1,\,t)=0, \quad t>0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad 0<x<1,\\ u_t(x,t-\tau)=f^0(t-\tau), \quad 0<x<1,\,0<t<\tau, \end{cases}$$

where $\alpha_1,\,\alpha_2>0$ *and* $\tau>0$.

This problem enters in the above abstract setting with the following choice.

$$\begin{aligned} &H=L^2(0,\,1), V=H_0^1(0,1); U_1=U_2=L^2(0,\,1),\\ &A:H^2(0,\,1)\cap H_0^1(0,1)\to H\ :\ \varphi\mapsto -\frac{d^2}{dx^2}\varphi,\\ &B_i:U_i=H\to H\subset V'\ :\ u\mapsto\sqrt{\alpha_i}u, i=1,2, \end{aligned}$$

with

$$(u,v)_H=\int_0^1 u(x)v(x)\,dx, \quad \forall u,v\in L^2(0,\,1).$$

Note that $B_i^*=B_i$.

A is a positive selfadjoint operator since it is the Friedrichs extension of the triple (H,V,a), where

$$a(u,v)=\int_0^1 u_x(x)v_x(x)\,dx, \quad \forall u,v\in V=H_0^1(0,1)$$

which is coercive on V, due to Poincaré's inequality and A has a compact inverse since $H_0^1(0,1)\hookrightarrow_c L^2(0,1)$.

Example 1.2. — *(unbounded feedbacks). The wave equation with boundary feedbacks*

$$(1.3)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\, t > 0 \\ u(0,\, t) = 0, \quad t > 0 \\ \dfrac{\partial u}{\partial x}(1,t) + \alpha_1 \dfrac{\partial u}{\partial t}(1,\, t) + \alpha_2 \dfrac{\partial u}{\partial t}(1,\, t-\tau) = 0, \quad t > 0, \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), \quad 0 < x < 1, \\ u_t(\xi, t-\tau) = f^0(t-\tau), \quad 0 < t < \tau. \end{cases}$$

where $\alpha_1,\, \alpha_2 > 0$ *and* $\tau > 0$.

The second example enters in the abstract setting with

$$\begin{aligned} &H = L^2(0,\, 1), V = \{u \in H^1(0,1) : u(0) = 0\}; \\ &A : \{u \in H^2(0,1) : u(0) = u_x(1) = 0\} \to H \,:\, \varphi \mapsto -\frac{d^2}{dx^2}\varphi, \\ &U_1 = U_2 = \mathbb{R}, \\ &B_i : \mathbb{R} \to V' \,:\, k \mapsto \sqrt{\alpha_i} k \delta_1, i = 1, 2. \end{aligned}$$

Again A is a positive selfadjoint operator since it is the Friedrichs extension of the triple (H, V, a), where the bilinear form a is defined as before. Again A has a compact inverse since $V \hookrightarrow_c L^2(0,1)$. Note that $B_i^* : V \to \mathbb{R} : u \mapsto \sqrt{\alpha_i} u(1), i = 1, 2$, because

$$\langle B_i k, v\rangle_{V'-V} = \sqrt{\alpha_i} k v(1) = k B_i^* v.$$

Example 1.3. — *The wave equation with internal unbounded feedbacks*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \alpha_1 \dfrac{\partial u}{\partial t}(\xi,\, t)\delta_\xi + \alpha_2 \dfrac{\partial u}{\partial t}(\xi,\, t-\tau)\delta_\xi = 0, \quad 0 < x < 1,\, t > 0 \\ u(0,\, t) = u(1,\, t) = 0, \quad t > 0 \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), \quad 0 < x < 1, \end{cases}$$

where $\xi \in (0,\, 1)$, $\alpha_1,\, \alpha_2 > 0$ *and* $\tau > 0$. *Again this example enters in the abstract setting when*

$$\begin{aligned} &H = L^2(0,\, 1), A : H^2(0,\, 1) \cap H_0^1(0,1) \to H \,:\, \varphi \mapsto -\frac{d^2}{dx^2}\varphi, \\ &V = H_0^1(0,1); U_1 = U_2 = \mathbb{R}, \\ &B_i : \mathbb{R} \to V' \,:\, k \mapsto \sqrt{\alpha_i} k \delta_\xi, i = 1, 2. \end{aligned}$$

Some instabilities. — If $\alpha_2 > \alpha_1$, the previous systems may be unstable, see [**4, 8, 9**]. We here check this property for Example 1.2. Hence some conditions between B_1 and B_2 have to be imposed to get stability.

Instabilities for Example 1.2. — We look for a solution u of (1.3) in the form

$$u(x,t) = e^{\lambda t}\varphi(x).$$

Hence (1.3) holds if and only if φ is solution of

$$\varphi_{xx} - \lambda^2\varphi = 0 \text{ in } (0,1)$$

$$\varphi(0) = 0,\ \varphi_x(1) + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\lambda\varphi(1) = 0.$$

Hence for $\lambda \neq 0$, φ is given by

$$\varphi(x) = C_1 \sinh(\lambda x).$$

Then the boundary condition at 1 will be satisfied if and only if

$$C_1\lambda(\cosh\lambda + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh\lambda) = 0.$$

Since we are looking for non trivial solutions, this is equivalent to

$$\cosh\lambda + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh\lambda = 0. \tag{1.4}$$

Lemma 1.1. — *If $\alpha_2 \geq \alpha_1 \geq 0$, there exists a discrete set T of delays τ such that for any $\tau \in T$ the system (1.3) is unstable, i.e., it admits exponentially unstable solutions.*

Proof. — We take the delay in the form

$$\tau = \frac{2(2k+1)}{2n+1} < 1,$$

with $n, k \in \mathbb{N}$ and we look for roots of (1.4) in the form

$$\lambda = \eta + \imath\frac{(2n+1)\pi}{2},$$

where $\eta \in \mathbb{R}$ is the remaining unknown. As $e^{-\lambda\tau} = -e^{-\eta\tau}$, (1.4) holds if and only if

$$F(\eta) := e^{\eta} - e^{-\eta} + (\alpha_1 - \alpha_2 e^{-\eta\tau})(e^{\eta} + e^{-\eta}) = 0.$$

But we notice that

$$F(0) = 2(\alpha_1 - \alpha_2) \leq 0,$$

while

$$\lim_{\eta\to+\infty} F(\eta) = +\infty.$$

Hence by the intermediate value theorem, there exists $\eta \geq 0$ such that $F(\eta) = 0$. The system (1.3) is then unstable for the countable set of delays τ in the above form. □

In the whole paper the notation $A \lesssim B$ means the existence of positive constants C which are independent of A and B such that $A \leq CB$; the notation $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ simultaneously.