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CONNECTED COMPONENTS OF THE STRATA OF THE MODULI SPACES OF QUADRATIC DIFFERENTIALS

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ABSTRACT. – In two fundamental classical papers, Masur [14] and Veech [21] have independently proved that the Teichmüller geodesic flow acts ergodically on each connected component of each stratum of the moduli space of quadratic differentials. It is therefore interesting to have a classification of the ergodic components. Veech has proved that these strata are not necessarily connected. In a recent work [8], Kontsevich and Zorich have completely classified the components in the particular case where the quadratic differentials are given by the global square of Abelian differentials.

Here we are interested in the complementary case. In a previous paper [11], we have described some particular components, namely the *hyperelliptic* connected components and showed that some strata are non-connected. In this paper, we give the general classification theorem: up to four exceptional cases in low genera, the strata of meromorphic quadratic differentials are either connected, or have exactly two connected components where one component is hyperelliptic, the other not. This result was announced in the paper [11].

Our proof is based on a new approach of the so-called Jenkins-Strebel differential. We will present and use the notion of *generalized permutations*.

RÉSUMÉ. – Dans des travaux maintenant classiques, Masur [14] et Veech [21] ont démontré indépendamment que le flot géodésique de Teichmüller est ergodique sur chaque composante connexe de chaque strate de l'espace des modules des différentielles quadratiques. Il devient dès lors intéressant d'avoir une description de ces composantes ergodiques. Veech a montré que ces strates ne sont pas nécessairement connexes. Dans un article récent, Kontsevich et Zorich [8] donnent une description complète des composantes dans le cas particulier où les différentielles quadratiques sont données par le carré de différentielles abéliennes.

Dans cet article, nous considérons le cas complémentaire. Dans un précédent article [11], nous montrions que les strates ne sont pas forcément connexes. Nous donnions une série de strates non-connexes possédant des composantes connexes *hyperelliptiques*. Dans cet article, nous démontrons le théorème général annoncé dans [11] : excepté quatre cas particuliers en petits genres, les strates de l'espace des modules des différentielles quadratiques ont au plus deux composantes connexes, les cas de non-connexité étant décrits exactement par [11] : une composante est hyperelliptique, l'autre non.

Notre preuve repose principalement sur une nouvelle approche des différentielles quadratiques de type Jenkins-Strebel, à savoir la notion de *permutations généralisées*.

1. Introduction

The moduli space of a genus g compact connected Riemann surface $\mathcal{S}$ endowed with an integrable meromorphic quadratic differential q is a disjoint union $\mathcal{H}_g \sqcup \mathcal{Q}_g$, where the isomorphism class of $(\mathcal{S}, q)$ belongs to $\mathcal{H}_g$ if and only if q is the (global) square of a holomorphic Abelian differential. It can be identified with the cotangent bundle of the moduli space $\mathcal{M}_g$ of compact connected smooth complex curves (see e.g. [5]). It carries a natural flow, called the Teichmüller geodesic flow (see [14, 21]). It has a natural stratification, whose strata are denoted by $\mathcal{H}(k_1, \cdots, k_n) = \mathcal{H}_g(k_1, \cdots, k_n)$ contained in $\mathcal{H}_g$ and $\mathcal{Q}(k_1, \cdots, k_n) = \mathcal{Q}_g(k_1, \cdots, k_n)$ contained in $\mathcal{Q}_g$, where $k_1, \ldots, k_n$ is the (unordered) list of multiplicities of the zeroes and poles of the quadratic differentials. It is well known that the flow preserves this stratification and that each stratum carries a complex algebraic orbifold structure of complex dimension $2g+n-\varepsilon$ (here $\varepsilon = 1$ or 2 depending respectively of the strata of Abelian differential or quadratic differentials). Masur and Smillie [15] proved that all of these strata (corresponding to the multiplicities satisfying the Gauss-Bonnet condition), except four particular cases in low genera, are non-empty.

The aim of this paper is motivated by a fundamental theorem, independently proved by Masur and Veech [14], [21], which asserts that the Teichmüller geodesic flow acts ergodically on each connected component of each stratum (with respect to a finite measure equivalent to the Lebesgue measure).

Kontsevich and Zorich [8] have recently described the set of connected components for the strata of $\mathcal{H}_g$. In [11], using a construction developed in [8] (hyperelliptic components), we showed that some strata in $\mathcal{Q}_g$ are non-connected. More precisely, we presented three series of one discrete-parameter strata which are non connected; those strata have a connected component consisting of hyperelliptic curves equipped with a "hyperelliptic differential". This component is called a *hyperelliptic component*.

In this paper, we describe the set of connected components of any stratum of $\mathcal{Q}_g$. The general case stabilizes at genus 5 and corresponds to Theorem 1.1. Theorem 1.2 gives the remaining cases.

THEOREM 1.1. – *Let us fix $g \geq 5$. Each stratum of the moduli space $\mathcal{Q}_g$ having a hyperelliptic connected component has exactly two connected components: one is hyperelliptic — the other not; the detailed list is given in* [11] *(see also below).*

Any other stratum of the moduli space $\mathcal{Q}_g$ of quadratic differentials is non empty and connected.

In small genera, there are some exceptional cases coming from the geometry of genus one and genus two surfaces (respectively elliptic and hyperelliptic curves). There are also 4 mysterious cases which appear.

THEOREM 1.2. – *Let us fix $g \leq 4$. The connected components of the strata of the moduli space $\mathcal{Q}_g$ fall in the following description:*

- *In genera* 0 *and* 1*, except two strata that are empty, any stratum is non empty and connected.*
- *In genus* 2*, there are two empty strata. There are also two non-connected strata. For these two, one component is hyperelliptic, the other not. Any other stratum of $\mathcal{Q}_2$ is non empty connected.*
- *In genera* 3 *and* 4*, each stratum with a hyperelliptic connected component has exactly two connected components: one is hyperelliptic, the other not.*
- *There are* 4 *sporadic strata in genus* 3 *and* 4 *which are non-connected and which do not possess a hyperelliptic component.*
- *Any other stratum of $\mathcal{Q}_3$ and $\mathcal{Q}_4$ is non empty and connected.*

1.1. Precise formulation of the statements

In order to establish notations and to give a precise statement, we review basic notions concerning moduli spaces, Abelian differentials and quadratic differentials. There is an abundant literature on this subject; for more details and proofs see for instance [2], [3], [4], [5], [9], [8], [14], [19], [20], [21], [23], [22],.... For a nice survey see [16] or [24].

1.1.1. *Background.* – For $g \geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs $(\mathcal{S}, \omega)$ where $\mathcal{S}$ is a genus g (closed connected) Riemann surface and $\omega \in \Omega(\mathcal{S})$ a non-zero holomorphic $1-$form defined on $\mathcal{S}$. The term moduli space means that we identify the points $(\mathcal{S}, \omega)$ and $(\mathcal{S}', \omega')$ if there exists an analytic isomorphism $f : \mathcal{S} \to \mathcal{S}'$ such that $f^*\omega' = \omega$.

For $g \geq 0$, we also define the moduli space of quadratic differentials $\mathcal{Q}_g$ *which are not the global square of Abelian differentials* as the moduli space of pairs $(\mathcal{S}, q)$ where $\mathcal{S}$ is a genus g Riemann surface and q a non-zero meromorphic quadratic differential defined on $\mathcal{S}$ such that q is not the global square of any Abelian differential. In addition, we assume that q has at most simple poles, if any. This last condition guaranties that the area of $\mathcal{S}$ in terms of the metric determined by q is finite:

$$\int_{\mathcal{S}} |q| < \infty.$$

We will denote by $\mathcal{H}(k_1, \ldots, k_n)$ the subset of $\mathcal{H}_g$ consisting of (classes of) pairs $(\mathcal{S}, \omega)$ such that ω possesses exactly n zeroes on $\mathcal{S}$ with multiplicities $(k_1, \ldots, k_n)$. We will also denote by $\mathcal{Q}(k_1, \ldots, k_n)$ the subset of $\mathcal{Q}_g$ consisting of pairs $(\mathcal{S}, q)$ such that q possesses exactly n singularities on $\mathcal{S}$ with multiplicities $(k_1, \ldots, k_n)$, $k_i \geq -1$.

Note that the Gauss-Bonnet formula implies that the sum of the multiplicities $\sum k_i$ equals $2g-2$ in the case of $\mathcal{H}(k_1, \ldots, k_n)$ and $4g-4$ in the case of $\mathcal{Q}(k_1, \ldots, k_n)$. In Section 2.1.1, we will present Thurston's approach to these surfaces via the theory of measured foliations.

From these definitions, it is a well known part of the Teichmüller theory that these spaces are (Hausdorff) complex analytic, and in fact algebraic, spaces (see [2] for a nice description of the stratum $\mathcal{Q}(1, \ldots, 1)$; see also [5], [9], [23]). Basically, one can see that as follows. We first concentrate on the strata of the moduli spaces $\mathcal{H}_g$.

Let $(\mathcal{S}, \omega^2)$ be a representative of an element in $\mathcal{H}(k_1, \ldots, k_n)$, S its underlying topological surface, and $P_1, \ldots, P_n$ its singular points. Let us denote by $\mathrm{hol} = \mathrm{hol}_{(\mathcal{S},\omega)}$ the group morphism $H_1(S, \{P_1 \ldots, P_n\}, \mathbb{Z}) \to \mathbb{C}$ defined by $\mathrm{hol}([\gamma]) = \int_\gamma \omega$ for every 1-cycle γ in S relative to $\{P_1 \ldots, P_n\}$. Fix a basis $(\gamma_1, \ldots, \gamma_{2g+n-1})$ of the free abelian group $H_1(S, \{P_1 \ldots, P_n\}, \mathbb{Z})$. Any other element of $\mathcal{H}(k_1, \ldots, k_n)$ will be represented by an element having the same underlying surface and the same singular points. With these notations, the map

$$\Phi = \begin{pmatrix} \mathcal{H}(k_1, k_2, \ldots, k_n) \longrightarrow & H^1(S, \{P_1, \ldots, P_n\}, \mathbb{C}) \\ \mathcal{S}' \longmapsto & (\gamma_1, \ldots, \gamma_{2g+n-1}) \mapsto (hol_{\mathcal{S}'}(\gamma_1), \ldots, hol_{\mathcal{S}'}(\gamma_{2g+n-1})) \end{pmatrix}$$

is named the *period map* and is a local homeomorphism in a neighbourhood of $(\mathcal{S}, \omega^2)$. Therefore we get a locally one-to-one correspondence between the corresponding stratum of $\mathcal{H}_g$ and an open domain in the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C}) \simeq \mathbb{C}^{2g+n-1}$. The changes of coordinates are affine maps outside the singularities of $\mathcal{H}(k_1, k_2, \ldots, k_n)$ and produce after a study of the singularities a differentiable orbifold structure on the strata of $\mathcal{H}_g$.

Let us now consider the case of a stratum of the moduli space $\mathcal{Q}_g$. For every $(\mathcal{S}, q) \in \mathcal{Q}_g$, consider the canonical double cover $\pi : \hat{\mathcal{S}} \to \mathcal{S}$ such that $\pi^* q = \omega^2$ for some holomorphic Abelian differential ω on $\hat{\mathcal{S}}$ (see for instance [12]). As above, we consider the period map between a neighborhood of the point $(\hat{\mathcal{S}}, \omega)$ and an open subset of $H^1(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_n\}; \mathbb{C})$. The covering involution $\tau : \hat{\mathcal{S}} \to \hat{\mathcal{S}}$ induces an involutive linear map on this cohomology vector space. Therefore, this vector space decomposes in two eigenspaces for τ^*, say E_{-1} and E_{+1}, with eigenvectors -1 and $+1$. Abelian differentials in E_{-1} are precisely those which arise from quadratic differentials on $\mathcal{S}$ by pull-back by π. Hence we obtain a one-to-one correspondence between a neighborhood of any point in the corresponding strata of $\mathcal{Q}_g$ and an open domain of $E_{-1} \simeq \mathbb{C}^{2g+n-2}$.

We next recall the construction of a measure μ defined on each stratum. For that, the tangent space to $\mathcal{H}_g$ (respectively $\mathcal{Q}_g$) at each point contains a lattice:

$$H^1(\mathcal{S}, \{P_1, \ldots, P_n\}; \mathbb{Z}) \oplus i \cdot H^1(\mathcal{S}, \{P_1, \ldots, P_n\}; \mathbb{Z}) \subset H^1(\mathcal{S}, \{P_1, \ldots, P_n\}; \mathbb{C}).$$

We define a measure μ on each stratum by pulling back the Lebesgue measure defined on $H^1(\mathcal{S}, \{P_1, \ldots, P_n\}; \mathbb{C})$ normalized so that the volume of the unit cube is 1.