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The Cauchy Problem for Wave Equations with non Lipschitz Coefficients

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THE CAUCHY PROBLEM FOR WAVE EQUATIONS WITH NON LIPSCHITZ COEFFICIENTS; APPLICATION TO CONTINUATION OF SOLUTIONS OF SOME NONLINEAR WAVE EQUATIONS

BY Ferruccio COLOMBINI AND Guy MÉTIVIER

ABSTRACT. – In this paper we study the Cauchy problem for second order strictly hyperbolic operators of the form

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\left(a_{j,k}\partial_{y_k}u\right) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f,$$

when the coefficients of the principal part are not Lipschitz continuous, but only "Log-Lipschitz" with respect to all the variables. This class of equation is invariant under changes of variables and therefore suitable for a local analysis. In particular, we show local existence, local uniqueness and finite speed of propagation for the noncharacteristic Cauchy problem. This provides an invariant version of a previous paper of the first author with N. Lerner [6]. We also give an application of the method to a continuation theorem for nonlinear wave equations where the coefficients above depend on u: the smooth solution can be extended as long as it remains Log-Lipschitz.

RÉSUMÉ. – On considère le problème de Cauchy pour des équations d'onde strictement hyperboliques :

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\left(a_{j,k}\partial_{y_k}u\right) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f,$$

quand les coefficients de la partie principale sont seulement "Log-Lipschitz" en toutes les variables. Cette classe d'équation est invariante par changement de variables et est donc une classe naturelle pour une étude locale intrinsèque. En particulier, on montre l'existence locale, l'unicité locale et la vitesse finie de propagation pour le problème de Cauchy non caractéristique, donnant une version invariante d'un résultat antérieur du premier auteur avec N. Lerner [6]. Pour les équations non linéaires où les coefficients ci-dessus dépendent de u, la méthode d'estimations permet de montrer que les solutions régulières se prolongent en solutions régulières aussi longtemps qu'elles restent Log-Lipschitz.

1. Introduction

In this paper we study the well-posedness of the Cauchy problem for second order strictly hyperbolic equations whose coefficients are not Lipschitz continuous:

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\left(a_{j,k}\partial_{y_k}u\right) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f. \tag{1.1}$$

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In Section 6, we will present an application of the methods developed for the analysis of the Cauchy problem to nonlinear wave equations, where the various coefficients above depend on u. It is known that the smooth solution can be extended as long as they remain Lipschitz continuous. We prove that this condition can be weakened, and that smooth solutions remain smooth as long as they remain Log-Lipschitz. We refer to Section 6 for a precise result and focus now on the analysis of the Cauchy problem.

The question of the well-posedness of the Cauchy problem for the wave equation with nonsmooth coefficients has already been studied in the case that the second order part has the special form, in coordinates $y = (t, x)$:

$$\partial_t^2 - \sum_{j,k=1}^{n} \partial_{x_j}\big(a_{j,k}\partial_{x_k} u\big) \tag{1.2}$$

and the Cauchy data are given on the space-like hyperplane $\{t = 0\}$. In this case, when the coefficients depend only on the time variable t, F. Colombini, E. De Giorgi and S. Spagnolo ([5]) have proved that the Cauchy problem is in general ill-posed in C^∞ when the coefficients are only Hölder continuous of order $\alpha < 1$, but is well-posed in appropriate Gevrey spaces. This has been extended to the case where the coefficients are Hölder in time and Gevrey in x ([14, 8]). Moreover, it is also proved in [5] that the Cauchy problem is well posed in C^∞ when the coefficients, which depend only on time, are "Log-Lipschitz" (in short LL) : recall that a function a of variables y is said to be LL on a domain Ω if there is a constant C such that

$$|a(y) - a(y')| \le C|y - y'| \Big(1 + \big|\mathrm{Log}|y - y'|\big|\Big) \tag{1.3}$$

for all y and y' in Ω. In [5], it is proved that for LL coefficients depending only on t and for initial data in the Sobolev spaces $H^s \times H^{s-1}$, the solution satisfies

$$u(t,\cdot) \in H^{s-\lambda t}, \quad \partial_t u(t,\cdot) \in H^{s-1-\lambda t} \tag{1.4}$$

with λ depending only on the LL norms of the coefficients and the constants of hyperbolicity. In particular, there is a loss of smoothness as time evolves and this loss does occur in general when the coefficients are not Lipschitz continuous, and is sharp, as shown in [3].

The analysis of the C^∞ well-posedness has been extended by F. Colombini and N. Lerner ([6]) to the case of equations, still with principal part (1.2), whose coefficients also depend on the space variables x. They show that the Cauchy problem is well-posed if the coefficients are LL in time and C^∞ in x. They also study the problem under the natural assumption of isotropic LL smoothness in (t, x). In this case one has to multiply LL functions with distributions in H^s. This is well defined only when $|s| < 1$. Therefore, one considers initial data in $H^s \times H^{s-1}$ with $0 < s < 1$, noticing that further smoothness would not help. Next, the loss of smoothness (1.4) forces us to limit t to an interval where $0 < s - \lambda t$, yielding only local in time existence theorems. We also refer to [6] for further discussions on the sharpness of LL smoothness.

However, the local uniqueness of the Cauchy problem and the finite speed of propagation for local solutions are not proved in [6]. The main goal of this paper is to address these questions. Classical methods, such as convexification, lead one to consider general equations (1.1) with LL coefficients in all variables. However, the meaning of the Cauchy problem for such equations is not completely obvious: as mentioned above, the maximal expected smoothness of the solutions is H^s with $s < 1$ and their traces on the initial manifold are not immediately defined. More importantly, in the general theory of smooth operators, the traces are defined using partial regularity results in the normal direction; in our case, the limited smoothness of the coefficients is a source of difficulties. It turns out that when $s \leq \frac{1}{2}$, one cannot in general define the traces of all the first order derivatives of u, but only the Neumann trace relative to the operator, using a weak formulation of the traces.

ASSUMPTION 1.1. – *L is a second order operator of the form* (1.1) *on a neighborhood Ω of $\underline{y}$, with coefficients $a_{j,k} \in LL(\Omega)$, b_j and c_j in $C^\alpha(\Omega)$, for some $\alpha \in]\frac{1}{2}, 1[$ and $d \in L^\infty(\Omega)$. Σ is a smooth hypersurface through $\underline{y}$ and L is strictly hyperbolic in the direction conormal to Σ.*

Shrinking Ω if necessary, we assume that Σ is defined by the equation $\{\varphi = 0\}$ with φ smooth and $d\varphi \neq 0$. We consider the one-sided Cauchy problem, say on the component $\Omega_+ = \Omega \cap \{\varphi > 0\}$. We use the Sobolev spaces $H^s(\Omega \cap \{\varphi > 0\})$ for $s \in \mathbb{R}$. As usual, we say that $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$, if for any relatively compact open subset Ω_1 of Ω, the restriction of u to $\Omega_1 \cap \{\varphi > 0\}$ belongs to $H^s(\Omega \cap \{\varphi > 0\})$. Similarly, $u \in H^s_{\text{comp}}(\Omega \cap \{\varphi \geq 0\})$ if $u \in H^s(\Omega \cap \{\varphi > 0\})$ has compact support in $\Omega \cap \{\varphi \geq 0\}$.

The adjoint operator

$$L^* v := \sum_{j,k=0}^{n} \partial_{y_k}\left(\overline{a}_{j,k}\partial_{y_j} v\right) - \sum_{j=0}^{n}\{\overline{c}_j \partial_{y_j} v + \partial_{y_j}(\overline{b}_j v)\} + \overline{d} v \tag{1.5}$$

has the same form as L. For u and v smooth, v compactly supported in $\Omega \cap \{\varphi \geq 0\}$, one has the (formal) identity

$$\big(Lu, v\big)_{L^2(\Omega_+)} - \big(u, L^* v\big)_{L^2(\Omega_+)} = \big(N_\nu u, v\big)_{L^2(\Sigma)} - \big(u, N'_\nu v\big)_{L^2(\Sigma)} \tag{1.6}$$

where

$$\begin{aligned} N_\nu u &= \sum_{j,k} \nu_k (a_{j,k}\partial_j u)_{|\Sigma}, \\ N'_\nu v &= \sum_{j,k} \nu_j (\overline{a}_{j,k}\partial_k v)_{|\Sigma} - \sum_j \nu_j\big((\overline{b}_j + \overline{c}_j) v\big)_{|\Sigma} \end{aligned} \tag{1.7}$$

and $\nu = (\nu_0, \ldots, \nu_d) \neq 0$ is conormal to Σ and the d-integration form on Σ is chosen accordingly.

LEMMA 1.2. – i) *For all $s \in]1-\alpha, 1+\alpha[$ and $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$, all the terms entering in the definition of Lu and $L^* u$ are well defined as distributions in $H^{s-2}_{loc}(\Omega \cap \{\varphi \geq 0\})$.*

ii) *For all $s \in]\frac{3}{2}, 1+\alpha[$ and $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$, the traces $N_\nu u$ and $N'_\nu u$ are well defined in $H^{s-\frac{3}{2}}_{loc}(\Sigma \cap \Omega)$.*

Proof. – This is due to multiplicative properties (see [6] and Corollary 3.6):

– If $\sigma \in]-1, 1[$, $a \in LL(\Omega)$ and $v \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$, then $av \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$.
– If $\sigma \in]-\alpha, \alpha[$, $a \in C^\alpha(\Omega)$ and $v \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$, then $av \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$.

□

Next, we recall that the subspace of functions with compact support in Ω_+ is dense in $H^\sigma(\Omega_+)$ when $|\sigma| < \frac{1}{2}$; moreover, for $0 \leq \sigma < \frac{1}{2}$ and for $u \in H^\sigma(\Omega_+)$ the pairing $(u, v)_{L^2(\Omega)}$ for $v \in L^2$ extends as the duality $\langle u, v\rangle_{H^\sigma \times H^{-\sigma}}$. With this remark in mind, the identity (1.6) holds for smooth functions:

LEMMA 1.3. – *For $s \in]\frac{3}{2}, 1+\alpha[$, $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$ and $v \in H^s_{\mathrm{comp}}(\Omega \cap \{\varphi \geq 0\})$, there holds*

$$\big\langle Lu, v\big\rangle_{H^{-\sigma}\times H^{\sigma}} - \big\langle u, L^* v\big\rangle_{H^{\sigma}\times H^{-\sigma}} = \big(N_\nu u, D_\Sigma v\big)_{L^2(\Sigma)} - \big(D_\Sigma u, N'_\nu v\big)_{L^2(\Sigma)} \tag{1.8}$$

with $\sigma = s - \frac{3}{2} \in]0, \frac{1}{2}[$ and $D_\Sigma u = u_{|\Sigma}$.

Proof. – It is sufficient to remark that for $\sigma \in [0, \frac{1}{2}[$, the Green's formula

$$\big\langle \partial_j u, v\big\rangle_{H^{-\sigma}\times H^{\sigma}} = -\big\langle u, \partial_j v\big\rangle_{H^{\sigma}\times H^{-\sigma}} + \big(\nu_j D_\Sigma u, D_\Sigma v\big)_{L^2(\Sigma)}$$

is satisfied for $u \in H^{1-\sigma}_{loc}(\Omega \cap \{\varphi \geq 0\})$ and $v \in H^{1-\sigma}_{\mathrm{comp}}(\Omega \cap \{\varphi \geq 0\})$. □

PROPOSITION 1.4. – *Let $D(L; H^s) = \{u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\}) : Lu \in L^2_{loc}(\Omega \cap \{\varphi \geq 0\})\}$. The operator N_Σ and D_Σ have unique extensions to $\bigcup_{s>1-\alpha} D(L; H^s)$ such that*

i) *For all $s \in]1-\alpha, \alpha[$, N_Σ (resp. D_Σ) is continuous from $D(L; H^s)$ into $H^{s-\frac{3}{2}}_{loc}(\Sigma \cap \Omega)$ (resp. $H^{s-\frac{1}{2}}_{loc}(\Sigma \cap \Omega)$).*

ii) *for all $s' \in]1-\alpha, \frac{1}{2}[$ such that $s' \leq s$ and all $v \in H^{2-s'}_{\mathrm{comp}}(\Omega \cap \{\varphi \geq 0\})$ there holds*

$$\big(Lu, v\big)_{L^2} - \big\langle u, L^* v\big\rangle_{H^{s'}\times H^{-s'}} = \big\langle N_\nu u, D_\Sigma v\big\rangle_{H^{s-\frac{3}{2}}\times H^{\frac{3}{2}-s}} - \big\langle D_\Sigma u, N'_\nu v\big\rangle_{H^{s-\frac{1}{2}}\times H^{\frac{1}{2}-s}}. \tag{1.9}$$

This proposition is proved in Section 5. Note that by Lemma 1.2, for $v \in H^{2-s'}_{\mathrm{comp}}$, $L^* v \in H^{-s'}_{\mathrm{comp}}$ and that $u \in H^{s'}_{loc}$ if $s' \leq s$. Moreover, $D_\Sigma v \in H^{\frac{3}{2}-s'}_{\mathrm{comp}} \subset H^{\frac{3}{2}-s}_{\mathrm{comp}}$ and $N'_\Sigma v \in H^{\frac{1}{2}-s'}_{\mathrm{comp}} \subset H^{\frac{1}{2}-s}_{\mathrm{comp}}$.

With this proposition, the Cauchy problem with source term in L^2 and solution in H^s, $s > 1-\alpha$, makes sense.

THEOREM 1.5 (Local existence). – *Consider $s > 1-\alpha$ and a neigborhood ω of $\underline{y}$ in Σ. Then there are $s' \in]1-\alpha, \alpha[$ and a neighborhood Ω' of $\underline{y}$ in $\mathbb{R}^{1+n}$ such that for all Cauchy data (u_0, u_1) in $H^s(\omega) \times H^{s-1}(\omega)$ near $\underline{y}$ and all $f \in L^2(\Omega' \cap \{\varphi > 0\})$ the Cauchy problem*

$$Lu = f, \quad D_\Sigma u = u_0, \quad N_\Sigma u = u_1, \tag{1.10}$$

has a solution $u \in H^{s'}(\Omega' \cap \{\varphi > 0\})$.

THEOREM 1.6 (Local uniqueness). – *If $s > 1-\alpha$ and $u \in H^s(\Omega \cap \{\varphi > 0\})$ satisfies*

$$Lu = 0, \quad D_\Sigma u = 0, \quad N_\Sigma u = 0, \tag{1.11}$$

then $u = 0$ on a neighborhood of $\underline{y}$ in $\Omega \cap \{\varphi \geq 0\}$.

REMARK 1.7. – If the coefficients of the first order term L_1 (see (2.3)) are also LL, the statements above are true with $\alpha = 1$ since the coefficients are then C^α for all $\alpha < 1$. If the b_j are C^α and the c_j are $C^{\tilde{\alpha}}$, the conditions are $1-\tilde{\alpha} < \alpha$ and the limitation on s is $1-\tilde{\alpha} < s$.