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SMALLNESS PROBLEM FOR QUANTUM AFFINE ALGEBRAS AND QUIVER VARIETIES

BY DAVID HERNANDEZ

ABSTRACT. – The geometric small property (Borho-MacPherson [2]) of projective morphisms implies a description of their singularities in terms of intersection homology. In this paper we solve the smallness problem raised by Nakajima [37, 35] for certain resolutions of quiver varieties [37] (analogs of the Springer resolution): for Kirillov-Reshetikhin modules of simply-laced quantum affine algebras, we characterize explicitly the Drinfeld polynomials corresponding to the small resolutions. We use an elimination theorem for monomials of Frenkel-Reshetikhin q-characters that we establish for non necessarily simply-laced quantum affine algebras. We also refine results of [21] and extend the main result to general simply-laced quantum affinizations, in particular to quantum toroidal algebras (double affine quantum algebras).

RÉSUMÉ. – La propriété géométrique de petitesse (Borho-MacPherson [2]) des morphismes projectifs implique une description de leurs singularités en termes d'homologie d'intersection. Dans cet article nous résolvons le problème de petitesse posé par Nakajima [37, 35] pour certaines résolutions de variétés carquois [37] (analogues de la résolution de Springer) : pour les modules de Kirillov-Reshetikhin des algèbres affines quantiques simplement lacées, nous caractérisons explicitement les polynômes de Drinfeld correspondant aux résolutions petites. Nous utilisons un théorème d'élimination pour les monômes des q-caractères de Frenkel-Reshetikhin, que nous établissons pour les algèbres affines quantiques non nécessairement simplement lacées. Nous raffinons également des résultats de [21] et étendons le résultat principal aux affinisées quantiques générales simplement lacées, en particulier aux algèbres toroïdales quantiques (algèbres quantiques doublement affines).

1. Introduction

Borho and MacPherson [2, Section 1.1] introduced remarkable geometric properties (smallness and semi-smallness) for a proper algebraic map $\pi : Z \to X$ where Z, X are irreducible complex algebraic varieties: for a finite stratification of X into irreducible smooth subvarieties, π is said to be semi-small if the dimension of the inverse image of a point in a

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stratum is at most half the codimension of the stratum, and π is said to be small if moreover the equality holds only if the stratum is dense. These properties do not depend on the stratification.

This geometric situation is of particular interest as the Beilinson-Bernstein-Deligne-Gabber decomposition Theorem [1] is simplified [2, Section 1.5] and provides an elegant description of the singularities of such maps in terms of intersection homology sheaves [15, 16]. A fundamental example of a semi-small morphism is given by the Springer resolution of the nilpotent cone of a complex simple Lie algebra, and the corresponding partial resolutions [2]. Nakajima [30, 31] defined important and intensively studied varieties called quiver varieties which depend on a quiver Q. They come with a resolution which is semi-small [31, Corollary 10.11] for a finite Dynkin diagram (see [34, Section 5.2]).

The graded version of quiver varieties is also of particular importance, for example for their deep relations with representations of quantum affine algebras (see [37]; the precise definition is reminded below). They also come with resolutions. A natural problem addressed in the present paper is to study the small property of these resolutions: we address [37, Conjecture 10.4] (see also [35]). Our study relies on the representation theory of quantum affine algebras: let us explain the context for our study.

In this paper $q \in \mathbb{C}^*$ is fixed and is not a root of unity. Affine Kac-Moody algebras $\hat{\mathfrak{g}}$ are infinite dimensional analogs of semi-simple Lie algebras $\mathfrak{g}$, and have remarkable applications in several branches of mathematics and physics (see [25]). Their quantizations $\mathcal{U}_q(\hat{\mathfrak{g}})$, called quantum affine algebras, have a very rich representation theory which has been intensively studied (see [7, 10] for references). In particular Drinfeld [12] discovered that they can also be realized as quantum affinization of usual quantum groups $\mathcal{U}_q(\mathfrak{g})$. By using this new realization, Chari-Pressley [7] classified their finite dimensional representations: they are parametrized by Drinfeld polynomials $(P_i(u))_{1\leq i\leq n}$ where n is the rank of $\mathfrak{g}$ and $P_i(u) \in \mathbb{C}[u]$ satisfies $P_i(0) = 1$.

A particular class of finite dimensional representations, called special modules, attracted much attention as Frenkel-Mukhin [13] proposed an algorithm which gives their q-character (analogs of usual characters adapted to the Drinfeld presentation of quantum affine algebras introduced by Frenkel-Reshetikhin [14]). Let us give some examples: for $k > 0, i \in I, a \in \mathbb{C}^*$, the Kirillov-Reshetikhin module $W_{k,a}^{(i)}$ is the simple module with Drinfeld polynomials

$$\begin{cases} P_j(u) = 1 \text{ for } j \neq i, \\ P_i(u) = (1 - uaq_i^{k-1})(1 - uaq_i^{k-3}) \cdots (1 - uaq_i^{1-k}). \end{cases}$$

(The q_i are certain powers of q, see section 3.) The $V_i(a) = W_{1,a}^{(i)}$ are called fundamental representations. The fundamental representations [13], and the Kirillov-Reshetikhin modules [36, 21] are special modules (this is the crucial point for the proof of the Kirillov-Reshetikhin conjecture). The corresponding standard module

$$M(X_{k,a}^{(i)}) = V_i(aq_i^{1-k}) \otimes V_i(aq_i^{3-k}) \otimes \cdots \otimes V_i(aq_i^{k-1})$$

is not special in general.

The breakthrough geometric approach of Nakajima [32, 37] to q-characters of representations of simply-laced quantum affine algebras via (graded) quiver varieties led to remarkable advances in the description of finite dimensional representations: for example this approach provides an algorithm [37] which computes the q-characters of *any* simple finite dimensional representations. Although in general the algorithm is very complicated, in some situations it provides a powerful tool to study these representations (for instance see [36]).

From the geometric point of view, the natural notion of small modules appeared in the following way: the small property of modules [37] is the representation theoretical interpretation of the smallness of certain resolutions of (graded) quiver varieties mentioned above.

A small module is special (but the converse is false in general). The representation theoretical interest of this notion is that all simple modules occurring in the Jordan-Hölder series of a small module are special, and so can be described by using the Frenkel-Mukhin algorithm.

A natural question is to characterize these small modules, and so the corresponding small resolutions. In particular, Nakajima ([37, Conjecture 10.4], [35]) raised the problem of characterizing the small standard modules corresponding to Kirillov-Reshetikhin modules.

In this paper we solve this problem by giving explicitly the corresponding Drinfeld polynomials.

The crucial point for our proof is an elimination theorem for monomials of q-characters, that we establish by refining our results of [21]. Indeed it is easy to produce monomials that occur in a certain q-character (for example see remark 3.16 below). But in general it is not clear if a given monomial *does not* occur in a q-character. The elimination theorem gives a criterion which implies that a monomial can be eliminated from the q-character of a simple module. Beyond the main result of the present paper (answer to the geometric smallness problem), we hope that this elimination theorem will be useful for other open problems in representation theory of quantum affine algebras. We already used it in a weak (non explicitly stated) form to prove the Kirillov-Reshetikhin conjecture [21]. Moreover it is used in [23] to study minimal affinizations of representations of quantum groups.

Let us state the main result of this paper. It can be stated in a simple compact way by using the following elementary definitions ($I = \{1, \dots, n\}$ is the set of vertices of the Dynkin diagram of $\mathfrak{g}$):

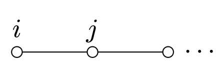
Definition 1.1. – *A node $i \in \{1, \dots, n\}$ is said to be extremal (resp. special) if there is a unique $j \in I$ (resp. three distinct $j, k, l \in I$) such that $C_{i,j} < 0$ (resp. $C_{i,j} < 0$, $C_{i,k} < 0$ and $C_{i,l} < 0$).*

For $i \in I$, we denote by d_i the minimal $d \geq 1$ such that there are distinct $i_1, \dots, i_d \in I$ satisfying $C_{i_j, i_{j+1}} < 0$ and i_d is special (if there are no special vertices, we set $d_i = +\infty$ for all $i \in I$).

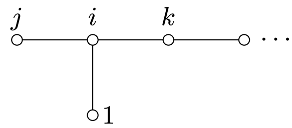
For example for $\mathfrak{g}$ of type A, we have $d_i = +\infty$ for all $i \in I$.

For an illustration, examples are given on the following pictures:

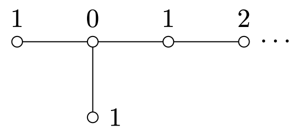
Extremal node i:



Special node i:



Distance d to a special node:



THEOREM 1.2 (Smallness problem). – *Let $k > 0, i \in I, a \in \mathbb{C}^*$. Then $M(X_{k,a}^{(i)})$ is small if and only if $k \leq 2$ or (i is extremal and $k \leq d_i + 1$).*

Remark: the condition is independent of the parameter $a \in \mathbb{C}^*$.

In particular for $\mathfrak{g} = sl_2$ or $\mathfrak{g} = sl_3$, all $M(X_{k,a}^{(i)})$ are small (it proves the corresponding [37, Conjecture 10.4]). In general it gives an explicit criterion so that the smallness holds. On the geometric side, it characterizes the small resolutions mentioned above.

Besides the statement of Theorem 1.2 is also satisfied for all simply-laced quantum affinizations $\mathcal{U}_q(\hat{\mathfrak{g}})$ ($\mathfrak{g}$ is an arbitrary simply-laced Kac-Moody algebra, not necessarily semi-simple), in particular for quantum toroidal algebras (double affine quantum algebras).

The general idea of the proof is first to establish the result for type A by using the elimination strategy of monomials explained above. We prove by induction on the highest weight that representations in a certain class are special. Then an argument allows to use the type A to prove the result for general types.

Let us describe the organization of this paper. In section 2 we explain the geometric context of our results. In section 3 we give some background on finite dimensional representations of quantum affine algebras and q-characters. In section 4 we recall from [37] the definition of small modules and the geometric characterization (Theorem 4.3). We refine a Theorem of [37] and give a more representation theoretical characterization (Theorem 4.8). However this last result is not enough to prove Theorem 1.2, and technical work is needed in the next sections. The first point is the (representation theoretical) elimination Theorem (Theorem 5.1) which is proved in section 5: it gives a condition which implies that a monomial *does not* appear in the q-character of a simple module. Additional technical results are also proved in section 5: in particular the notion of thin modules (with l-weight spaces of dimension 1) is introduced and studied. In section 6, we complete the proof of Theorem 1.2: first type A is discussed, and then the general case is treated. The proof of the result for general simply-laced quantum affinizations is also discussed.

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