ISSN 0012-9593

quatrième série - tome 41 fascicule 2 mars-avril 2008

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 41, 2008, p. 307 à 336

EQUIDISTRIBUTION TOWARDS THE GREEN CURRENT FOR HOLOMORPHIC MAPS

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ABSTRACT. – Let f be a non-invertible holomorphic endomorphism of a projective space and f^n its iterate of order n. We prove that the pull-back by f^n of a generic (in the Zariski sense) hypersurface, properly normalized, converges to the Green current associated to f when n tends to infinity. We also give an analogous result for the pull-back of positive closed $(1,1)$-currents and a similar result for regular polynomial automorphisms of $\mathbb{C}^k$.

RÉSUMÉ. – Soient f un endomorphisme holomorphe non-inversible d'un espace projectif et f^n son itéré d'ordre n. Nous prouvons que l'image réciproque par f^n d'une hypersurface générique (au sens de Zariski), proprement normalisée, converge vers le courant de Green associé à f quand n tend vers l'infini. Nous donnons également un résultat analogue pour les images réciproques des $(1,1)$-courants positifs fermés et un résultat similaire pour les automorphismes polynomiaux réguliers de $\mathbb{C}^k$.

1. Introduction

Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the projective space $\mathbb{P}^k$. Let ω denote the Fubini-Study form on $\mathbb{P}^k$ normalized so that ω is cohomologous to a hyperplane or equivalently $\int_{\mathbb{P}^k} \omega^k = 1$. It is well-known that the sequence of smooth positive closed $(1,1)$-forms $d^{-n}(f^n)^*(\omega)$ converges weakly to a positive closed $(1,1)$-current T of mass 1. Moreover, T has locally continuous potentials and is *totally invariant*, i.e. $f^*(T) = dT$. We call T *the Green current* of f. The complement of the support of T is the Fatou set, i.e. the sequence (f^n) is locally equicontinuous there. We refer the reader to the survey [29] for background. Our main results in this paper are the following theorems, where $[\cdot]$ denotes the current of integration on a complex variety.

THEOREM 1.1. – *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ and T the Green current associated to f. There is a proper analytic subset $\mathscr{E}$ of $\mathbb{P}^k$ such that if H is a hypersurface of degree s in $\mathbb{P}^k$ which does not contain any irreducible component of $\mathscr{E}$*

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then $d^{-n}(f^n)^*[H]$ converge to sT in the sense of currents when n tends to infinity. Moreover, $\mathscr{E}$ is totally invariant, i.e. $f^{-1}(\mathscr{E}) = f(\mathscr{E}) = \mathscr{E}$.

The exceptional set $\mathscr{E}$ will be explicitly constructed in Sections 6 and 7. It is the union of totally invariant proper analytic subsets of $\mathbb{P}^k$ which are minimal. That is, they have no proper analytic subsets which are totally invariant, see Example 7.5. That example shows that $\mathscr{E}$ is not the maximal totally invariant analytic set. The previous result is in fact a consequence of the following one, see also Theorem 7.1 for a uniform convergence result.

THEOREM 1.2. – *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ and T the Green current associated to f. There is a proper analytic subset $\mathscr{E}$ of $\mathbb{P}^k$, totally invariant, such that if S is a positive closed $(1,1)$-current of mass 1 in $\mathbb{P}^k$ whose local potentials are not identically $-\infty$ on any irreducible component of $\mathscr{E}$ then $d^{-n}(f^n)^*(S) \to T$ as $n \to \infty$.*

The space $\mathcal{H}_d$ of holomorphic maps f of a given algebraic degree $d \geq 2$ is an irreducible quasi-projective manifold. We will also deduce from our study the following result due to Fornæss and the second author [18], see also [16, 29].

THEOREM 1.3. – *There is a dense Zariski open subset $\mathcal{H}_d^*$ of $\mathcal{H}_d$ such that if f is a map in $\mathcal{H}_d^*$ then $d^{-n}(f^n)^*(S_n) \to T$ for every sequence (S_n) of positive closed $(1,1)$-currents of mass 1 in $\mathbb{P}^k$.*

The rough idea in order to prove our main results is as follows. Write $S = dd^c u + T$. Then, the invariance of T implies that $d^{-n}(f^n)^*(S) = d^{-n}dd^c(u\circ f^n) + T$. We have to show, in different situations, that $d^{-n}u\circ f^n$ converge to 0 in L^1. This implies that $d^{-n}(f^n)^*(S) \to T$. So, we have to study the asymptotic contraction (*à la* Lojasiewicz) by f^n. The main estimate is obtained using geometric estimates and convergence results for plurisubharmonic functions, see Theorem 5.1. If $d^{-n}u \circ f^n$ do not converge to 0, then using that the possible contraction is limited, we construct a limit v with strictly positive Lelong numbers. We then construct other functions w_{-n} such that the current $dd^c w_{-n} + T$ has Lelong numbers $\geq \alpha_0 > 0$ and $w_0 = d^{-n} w_{-n} \circ f^n$. It follows from the last identity that w_0 has positive Lelong numbers on an infinite union of analytic sets of a suitable dimension. The volume growth of these sets implies that the current associated to w_0 has too large self-intersection. This contradicts bounds due to Demailly and Méo [5, 26]. (One should notice that the Demailly-Méo estimates depend on the L^2 estimates for the $\overline{\partial}$-equation; they were recently extended to the case of compact Kähler manifolds by Vigny [33].) The previous argument has to be applied inductively on totally invariant sets for f, which are a priori singular and on which we inductively show the convergence to 0, starting with sets of dimension 0. So, we also have to develop the basics of the theory of weakly plurisubharmonic functions on singular analytic sets which is probably of independent interest. The advantage of this class of functions is that it has good compactness properties.

One may conjecture that totally invariant analytic sets should be unions of linear subspaces of $\mathbb{P}^k$. The case of dimension $k = 2$ is proved in [3, 28]. These authors complete the result in [17]. If this were true for $k \geq 3$, our proof would be technically simpler. It is anyway interesting to carry the analysis without any assumption on the totally invariant sets since our approach may be extended to the case of meromorphic maps on compact Kähler manifolds. At the end of the paper, we will consider the case of regular polynomial automorphisms of $\mathbb{C}^k$.

The problem of convergence was first considered by Brolin for polynomials in dimension 1 and then by Lyubich, Freire, Lopes and Mañé for rational maps in $\mathbb{P}^1$ [2, 19, 24]. In dimension $k = 2$, Fornæss and the second author proved that $\mathscr{E}$ is empty when the local multiplicity of f at every point is $\leq d-1$, see [18]. This implies Theorem 1.3 in dimension 2 for $S_n = S$. The proof in [18] can be extended to the general case, see also [29].

The family of hyperplanes in $\mathbb{P}^k$ is parametrized by a projective space of dimension k. It follows from Theorem 1.1 that for a hyperplane H, generic in the Zariski sense, we have $d^{-n}(f^n)^*[H] \to T$. Russakovskii and Shiffman have proved this result for H out of a pluripolar set in the space of parameters [27]. Analogous results for subvarieties in arbitrary Kähler manifolds were proved by the authors in [10]. Concerning Theorems 1.1 and 1.2, our conditions are not optimal. Indeed, it might happen that the potentials of S are identically $-\infty$ on some components of $\mathscr{E}$ and still $d^{-n}(f^n)^*(S) \to T$.

In the case of dimension $k = 2$, our results (except several uniform convergences, e.g. Theorem 7.1) can be deduced from results by Favre and Jonsson. These authors say that their condition is necessary and sufficient in order to have the previous convergence, see [13], and they give needed tools for the proof in [14, p.310]. In which case, if the Lelong number of S vanishes at generic points on each irreducible component of an exceptional set then $d^{-n}(f^n)^*(S) \to T$. The problem is still open in higher dimension. When the Lelong number of S is 0 at every point of $\mathbb{P}^k$, the convergence $d^{-n}(f^n)^*(S) \to T$ was obtained by Guedj [20], see also Corollary 5.9 below. In these works, the problem of convergence is reduced to the study of sizes of images of balls under iterates of f. This approach was first used in [16, 18].

Recall that the self-intersection $T^p := T \wedge \cdots \wedge T$, p times, defines a positive closed (p, p)-current which is totally invariant, i.e. $f^*(T^p) = d^p T^p$, see [11] for the pull-back operator on currents. It is natural to consider the analogous equidistribution problem towards T^p.

CONJECTURE 1.4. – *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and T its Green current. Then $d^{-pn}(f^n)^*[H]$ converge to sT^p for every analytic subset H of $\mathbb{P}^k$ of pure codimension p and of degree s which is generic. Here, H is generic if either $H \cap E = \varnothing$ or* $\operatorname{codim} H \cap E = p + \operatorname{codim} E$ *for any irreducible component E of every totally invariant analytic subset of $\mathbb{P}^k$.*

We will see later that there are only finitely many analytic sets which are totally invariant. Theorem 1.1 proves the conjecture for $p = 1$. Indeed, in that case, it is equivalent to check the condition for minimal totally invariant analytic sets. For $p = k$, the measure $\mu := T^k$ is the unique invariant measure of maximal entropy, see [18, 1, 29]. In this case, the conjecture was proved by the authors in [9]. Weaker results in this direction were obtained in [18] and [1]. We will give some details in Theorem 6.6. For $2 \leq p \leq k-1$, the authors have proved in [12] that for f in a Zariski dense open set $\mathcal{H}'_d \subset \mathcal{H}_d$, there is no proper analytic subset of $\mathbb{P}^k$ which is totally invariant and that the conjecture holds. Indeed, a version of Theorem 1.3 is proved.

2. Plurisubharmonic functions

We refer the reader to [22, 6, 10] for the basic properties of plurisubharmonic (psh for short) and quasi-psh functions on smooth manifolds. In order to study the Levi problem for analytic spaces X, the psh functions which are considered, are the restrictions of psh functions on an open set of $\mathbb{C}^k$ for a local embedding of X. Let $u : X \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function which is not identically equal to $-\infty$ on any irreducible component of X. Fornæss-Narasimhan proved that if u is subharmonic or equal to $-\infty$ on any holomorphic disc in X, then u is psh in the above sense [15]. However, this class does not satisfy good compactness properties which are useful in our analysis. Assume that X is an analytic space of pure dimension p. Let $\operatorname{reg}(X)$ and $\operatorname{sing}(X)$ denote the regular and the singular parts of X. We consider the following weaker notion of psh functions which is modeled after the notion of weakly holomorphic functions. The class has good compactness properties.

Definition 2.1. – A function $v : X \to \mathbb{R} \cup \{-\infty\}$ is *wpsh* if

(a) v is psh on $X \setminus \operatorname{sing}(X)$;
(b) for $a \in \operatorname{sing}(X)$, $v(a) = \limsup v(x)$ with $x \in \operatorname{reg}(X)$ and $x \to a$.

Fornæss-Narasimhan's theorem implies that psh functions are wpsh. Wpsh functions are psh when X is smooth. One should notice that the restriction of a wpsh function to an irreducible component of X is not necessarily wpsh. For example, consider $X = \{xy = 0\}$ in the unit ball of $\mathbb{C}^2$, let $v = 0$ on $\{x = 0\} \setminus (0, 0)$ and $v = 1$ on $\{y = 0\}$, then v is wpsh on X but its restriction to $\{x = 0\}$ is not wpsh. Consider the (strongly) psh function $v_n := |x|^{1/n}$ on X. The sequence v_n converges to v in $L^1(X)$. So, psh functions on analytic sets do not have good compactness properties.

Proposition 2.2. – *Let $Z \subset X$ be an analytic subset of dimension $\leq p - 1$ and v' a wpsh function on $X \setminus Z$. If v' is locally bounded from above near Z then there is a unique wpsh function v on X equal to v' outside Z.*

Proof. – The extension to a psh function on $\operatorname{reg}(X)$ is well-known. So, we can assume that $Z \subset \operatorname{sing}(X)$. Condition (b) in Definition 2.1 implies the uniqueness of the extension of v'. Define $v(a) := \limsup v(x)$ with $x \notin Z$ and $x \to a$. It is clear that $v = v'$ out of Z and v satisfies the conditions in Definition 2.1. □

Now assume for simplicity that X is an analytic subset of pure dimension p of an open set U in $\mathbb{C}^k$. The general case can be deduced from this one. The following results give characterizations of wpsh functions.

Proposition 2.3. – *Let $\pi : \widetilde{X} \to X \subset U$ be a desingularization of X. If v is a wpsh function on X then there is a psh function $\widetilde{v}$ on $\widetilde{X}$ such that $v(x) = \max_{\pi^{-1}(x)} \widetilde{v}$ for $x \in X$. Conversely, if $\widetilde{v}$ is psh on $\widetilde{X}$ then $x \mapsto \max_{\pi^{-1}(x)} \widetilde{v}$ defines a wpsh function on X.*