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Coarse topology, enlargeability, and essentialness

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COARSE TOPOLOGY, ENLARGEABILITY, AND ESSENTIALNESS

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ABSTRACT. – Using methods from coarse topology we show that fundamental classes of closed enlargeable manifolds map non-trivially both to the rational homology of their fundamental groups and to the K-theory of the corresponding reduced C^*-algebras. Our proofs do not depend on the Baum–Connes conjecture and provide independent confirmation for specific predictions derived from this conjecture.

RÉSUMÉ. – En utilisant des méthodes de topologie à grande échelle, on prouve que les classes fondamentales des variétés agrandissables ne s'annulent pas, ni dans l'homologie rationnelle de leurs groupes fondamentaux, ni dans la K-théorie des C^*-algèbres réduites correspondantes. Nos résultats ne dépendent pas de la conjecture de Baum–Connes, et confirment de façon indépendante certaines conséquences de cette conjecture.

1. Introduction and statement of results

In this paper we use methods from coarse topology to prove certain homological properties of enlargeable manifolds. The defining property of this class of manifolds is that they admit covering spaces that are uniformly large in all directions. The intuitive geometric meaning of enlargeability is naturally captured by concepts of coarse topology, in particular by the notion of macroscopic largeness. We proceed by showing that enlargeability implies macroscopic largeness, which in turn implies homological statements in classical, rather than coarse, algebraic topology.

Using completely different methods, related results were previously proved in [15, 16]. We shall discuss the comparison between the two approaches later in this introduction, after setting up some of the terminology to be used. Suffice it to say for now that our results here, unlike those of [15, 16], are relevant to the Baum–Connes conjecture for the reduced group C^*-algebra, in that we verify specific predictions derived from this conjecture.

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Enlargeability. – Several versions of the notion of enlargeability or hypersphericity were introduced by Gromov and Lawson in [13, 14]. Here is the basic definition:

Definition 1.1. – A closed oriented manifold M of dimension n is called *enlargeable* if for every $\epsilon > 0$ there is a covering space $M_\epsilon \longrightarrow M$ that admits an ϵ-contracting map

$$f_\epsilon \colon M_\epsilon \longrightarrow (S^n, g_{can})$$

to the n-sphere with its canonical metric, which is constant outside a compact set, and is of nonzero degree.

Here all covering spaces M_ϵ are given the pullback metrics induced by an arbitrary metric on M. The choice of metric on M matters only in that it has to be independent of ϵ.

A variation on this definition is obtained by restricting the kind of covering space allowed for M_ϵ. We shall call M *universally enlargeable* if it is enlargeable and for all ϵ the covering M_ϵ can be taken to be the universal covering $\widetilde{M} \longrightarrow M$. We shall call M *compactly enlargeable* if it is enlargeable and all M_ϵ can be taken to be compact, equivalently to be finite-sheeted coverings.

Essentialness. – Recall that Gromov [9] called a closed oriented manifold M *essential* if its fundamental class maps non-trivially to the rational homology of $B\pi_1(M)$ under the classifying map of its universal cover. It is natural to extend this definition to more general situations. For any homology theory E, we say that an E-oriented manifold M is E*-essential* if its orientation class maps non-trivially to $E_*(B\pi_1(M))$ under the classifying map of the universal covering.

In the context of coarse topology, one replaces the usual orientation class of M by the orientation class of the universal covering $\widetilde{M}$ in the coarse homology $HX_*(\widetilde{M})$, see Section 2 below. Passing to the coarse homology of the universal covering is a procedure not unlike passing from M to the classifying space of its fundamental group, and the coarse fundamental class $[\widetilde{M}]_X$ may well vanish. We shall say that a manifold M (or its universal covering) is *macroscopically large* if it is essential for coarse homology, i. e. if $[\widetilde{M}]_X \neq 0 \in HX_*(\widetilde{M})$. In fact, Gromov suggested various versions of macroscopic largeness in [10, 11, 12], and this definition, taken from [8], is just one particular way of formalizing the concept.

We can now state our first main result.

Theorem 1.2. – (1) *Universally enlargeable manifolds are macroscopically large.*

(2) *Macroscopically large manifolds are essential in rational homology.*

There are results by Dranishnikov [6] addressing the converse to the first part of this theorem. He has shown that other notions of macroscopic largeness sometimes imply versions of enlargeability.

Combining the two implications in Theorem 1.2, we obtain:

Corollary 1.3. – *Universally enlargeable manifolds are essential in rational homology.*

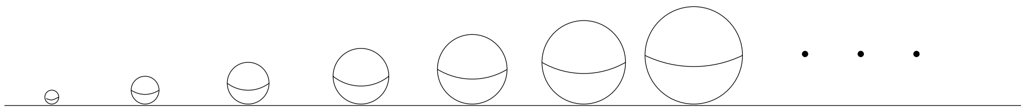

FIGURE 1. The connected balloon space B^n

That compactly enlargeable manifolds are essential was conjectured by Burghelea [26, Problem 11.1] quite some time ago and was proved fairly recently by Hanke and Schick [15], using index theory and the K-theory of C^*-algebras. One of the motivations for the present paper was the wish to give a direct and elementary proof of such a result, which does not use index theory and K-theory. After we achieved this goal by finding the proof of Theorem 1.2 given in Section 3 below, it turned out that the sophisticated methods of [15] can also be adapted to the consideration of infinite covers [16].

While the ideas involved in our proof of Theorem 1.2 are indeed geometric and elementary, they do fit naturally into the framework of coarse homology, which we recall in Section 2 following the books [22, 24]. Our argument makes essential use of the coarse space

$$B^n = [0,\infty) \bigcup_{\{1,2,3,\ldots\}} (\cup_i S^n(i)) \ .$$

This balloon space, sketched in Figure 1, is a coarse analogue of the one-point union. It is defined using a collection of n-spheres of increasing radii $i = 1, 2, 3, \ldots$, with the sphere of radius i attached to the point $i \in [0,\infty)$ at the south pole of S^n, and is equipped with the path metric.

The enlargeability assumption will be used to construct a coarse map

$$\widetilde{M} \longrightarrow B^n$$

that sends the coarse fundamental class of $\widetilde{M}$ to a nonzero class in the coarse homology of B^n (see Proposition 3.1). After this has been established, the proof of Theorem 1.2 can be completed quite easily.

Applications to the Baum–Connes map. – After giving the proof of Theorem 1.2, we proceed to use coarse topology to study the relation between enlargeability and the Baum–Connes assembly map in complex K-theory. This will lead us to some novel results on the Baum–Connes map that are interesting both in their own right and because of what they say about the relationship between various obstructions to the existence of positive scalar curvature metrics.

To formulate our results we make the following definition.

DEFINITION 1.4. – A closed K-theory oriented manifold M is *Baum–Connes essential* if the image of its K-theoretic fundamental class under the composite map

$$K_*(M) \xrightarrow{c_*} K_*(B\pi_1(M)) \xrightarrow{\mu} K_*(C^*_{\mathrm{red}}\pi_1(M))$$

is non-zero. Here, $c\colon M \longrightarrow B\pi_1(M)$ classifies the universal covering of M, and μ is the Baum–Connes assembly map.

In contrast to [15, 16], we will work with the *reduced* group C^*-algebra throughout the present paper. We use the letter K for the compactly supported complex K-homology defined by the K-theory spectrum. This is different from the convention in [22], where K_* denotes the analytically defined, hence locally finite K-homology.

Recall that a smooth manifold M is orientable with respect to K-theory if and only if its tangent bundle admits a Spin^c-structure. If M is compact, then any choice of Spin^c-structure determines a fundamental class $[M]$ in K-homology given by the corresponding Dirac operator, cf. [18, Chapter 11]. The image

$$\alpha(M) = \mu \circ c_*([M]) \in K_*(C^*_{\mathrm{red}}\pi_1(M))$$

is given by the index of the Spin^c Dirac operator on M twisted by the flat Hilbert module bundle

$$\widetilde{M} \times_{\pi_1(M)} C^*_{\mathrm{red}}\pi_1(M) \longrightarrow M$$

on M, as can be seen for example by a description of the Baum–Connes assembly map via Kasparov's KK-theory; cf. [3].

If a Spin^c-structure on M is induced by a spin structure, then the above construction can also be performed in real K-theory, leading to $\alpha_{\mathbb{R}}(M) \in KO_*(C^*_{\mathrm{red}}\pi_1(M))$. In this case $\alpha(M)$ is the image of $\alpha_{\mathbb{R}}(M)$ under complexification. The Weitzenböck formula for the spin Dirac operator implies via the Lichnerowicz argument that if M endowed with the fundamental class of a spin structure is Baum–Connes essential, then it does not admit a metric of positive scalar curvature. The Gromov–Lawson–Rosenberg conjecture predicts that the vanishing of $\alpha_{\mathbb{R}}(M)$ on a closed spin manifold M is not only necessary, but also sufficient for the existence of a positive scalar curvature metric on M. Although this conjecture does not hold in general [7, 25], it is expected that $\alpha_{\mathbb{R}}(M)$ captures all index-theoretic obstructions to the existence of a positive scalar curvature metric on M. This expectation is based in part on the relationship between the Gromov–Lawson–Rosenberg conjecture and the Baum–Connes conjecture.

Recall [2] that the Baum–Connes conjecture claims that for any discrete group Γ, the assembly map

$$K^{\Gamma}_*(\underline{E}\Gamma) \longrightarrow K_*(C^*_{\mathrm{red}}\Gamma)$$

is an isomorphism, where $\underline{E}\Gamma$ is the universal space for proper Γ-actions, and K_* denotes K-homology with compact supports. The assembly map

$$\mu\colon K_*(B\Gamma) \longrightarrow K_*(C^*_{\mathrm{red}}\Gamma)$$

considered above factors as

$$K_*(B\Gamma) \xrightarrow{\cong} K^{\Gamma}_*(E\Gamma) \xrightarrow{\gamma} K^{\Gamma}_*(\underline{E}\Gamma) \longrightarrow K_*(C^*_{\mathrm{red}}\Gamma)\ ,$$

where the first map is the canonical isomorphism between the equivariant K-theory of the free Γ-space $E\Gamma$ and the K-theory of the quotient $B\Gamma$ and γ is induced by the canonical map $E\Gamma \to \underline{E}\Gamma$. Stolz [27] has proved that if M is spin and the Baum–Connes conjecture holds for the group $\pi_1(M)$, then the vanishing of $\alpha_{\mathbb{R}}(M)$ is sufficient for M to stably admit a metric of positive scalar curvature. Here "stably" means that one allows the replacement of M by its product with many copies of a Bott manifold B, which is any simply connected